Differential Equations

Math 341 Spring 2005 — MWF 8:30 - 9:25am Fowler North 2

 — `http://faculty.oxy.edu/ron/math/341`

Class 23: **Monday March 28**

TITLE *The Matrix Exponential*
CURRENT READING Zill 8.4

Homework Set #9
Zill, Section 8.3: 3*, 9*, 11*, 19* *EXTRA CREDIT 32*
Zill, Section 8.4: 1*, 2*, 5*, 23* *EXTRA CREDIT 26*
Zill, Chapter 8 In Review: 3*, 4*, 11*, 15*

SUMMARY
One way of writing the solution of the homogeneous linear systems $\vec{x}' = A\vec{x}$ is $\vec{x} = e^{At}\vec{c}$.

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1. Matrix Exponential

DEFINITION: **matrix exponential**

For any $n \times n$ square matrix A, $\quad e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots + A^k\frac{t^k}{k!}$

The matrix e^{At} is a fundamental matrix; it has the property that $(e^{At})' = A(e^{At})$.

Exercise

Show that the solution of the single linear first-order differential equation $\frac{dx}{dt} = ax + f(t)$ is the sum of the homogeneous and nonhomogeneous solutions $x(t) = x_h + x_p$, in other words, $x(t) = ce^{at} + e^{at}\int_{t_0}^{t} e^{-as} f(s)ds$

Similarly, the solution to $\vec{x} = A\vec{x} + \vec{f}(t)$ can be written as $\vec{x} = \vec{x}_h + \vec{x}_p = e^{At}\vec{c} + e^{At}\int_{t_0}^{t} e^{-As}\vec{f}(s)ds$

RECALL

If one can diagonalize a matrix $A = SDS^{-1}$ where S is a matrix consisting of the eigenvectors of A and D is a diagonal matrix with the eigenvalues of A along the diagonal then $e^{At} = Se^{Dt}S^{-1}$

EXAMPLE Let's solve $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \vec{x}$ using the matrix exponential.

Introducing The Laplace Transform

Zill mentions that another way to use the matrix exponential is to solve the problem using the Laplace Transform. See Zill, Example 1 on page 362.

A Laplace Transform $\mathcal{L}$ is an operator which takes a function $F(t)$ as its input and produces $f(s)$ as its input. The Inverse Laplace Transform $\mathcal{L}^{-1}$ takes $f(s)$ as input and produces $F(t)$ as output. It turns out (we'll see why later!) that $\mathcal{L}[e^{At}] = (s\mathcal{I} - A)^{-1}$ which means that $\mathcal{L}^{-1}[(s\mathcal{I} - A)^{-1}] = e^{At}$ also.

Exercise Show that when $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, $(s\mathcal{I} - A)^{-1} = \begin{bmatrix} \dfrac{s+2}{s(s+1)} & \dfrac{-1}{s(s+1)} \\ \dfrac{2}{s(s+1)} & \dfrac{s-1}{s(s+1)} \end{bmatrix}$

Using partial fractions one can re-write this matrix as $(s\mathcal{I} - A)^{-1} = \begin{bmatrix} \dfrac{2}{s} - \dfrac{1}{s+1} & \dfrac{-1}{s} + \dfrac{1}{s+1} \\ \dfrac{2}{s} - \dfrac{2}{s+1} & -\dfrac{1}{s} + \dfrac{2}{s+1} \end{bmatrix}$

which when the Inverse Laplace Transform is applied,

$e^{At} = \mathcal{L}^{-1}[(s\mathcal{I} - A)^{-1}] = \begin{bmatrix} 2 - e^{-t} & -1 + e^{-t} \\ 2 - 2e^{-t} & -1 + 2e^{-t} \end{bmatrix}$