Differential Equations

Class 22: Friday March 25

TITLE *Non-homogeneous Systems of Linear Systems of First Order DEs*
CURRENT READING Zill 8.2, Edwards & Penney Handout

Homework Set #9
Zill, Section 8.3: 3*, 9*, 11*, 19*, *EXTRA CREDIT 32* (Use Variation of Parameters only)
Zill, Section 8.4: 1*, 2*, 5*, 23*, *EXTRA CREDIT 26*
Zill, Chapter 8 In Review: 3*, 4*, 11*, 15*

SUMMARY
We will apply the now-familiar technique of **the method of variation of parameters** to solve nonhomogeneous systems of DEs of the form $\vec{x}' = A\vec{x} + \vec{f}$.

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1. Complex Eigenvalues

If the eigenvalues of the matrix A are complex, then they will appear as complex conjugates (i.e. they will have the form $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$) where $i^2 = -1$ and α and β are both real numbers. The eigenvectors will also be complex. The corresponding solution to $\vec{x}' = A\vec{x}$ will be linear combinations of $\vec{X} = \vec{v}e^{\lambda t}$ and its conjugate $\vec{X}^* = \vec{v}^* e^{\lambda^* t}$.

RECALL $e^{i\theta} = \cos\theta + i\sin(\theta)$. One can also obtain real-valued solutions from these complex solutions by choosing complex versions for the constants. The general solution in this case will be a linear combination of $[\text{ Re }(\vec{v})\cos\text{ Im }(\lambda)t - \text{ Im }(\vec{v})\sin\text{ Im }(\lambda)t]e^{\text{ Re }(\lambda)t}$ and $[\text{ Im }(\vec{v})\cos\text{ Im }(\lambda)t + \text{ Re }(\vec{v})\sin\text{ Im }(\lambda)t]e^{\text{ Re }(\lambda)t}$

Exercise Solve the initial value problem $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2. Fundamental Matrix

RECALL We have said that we can write the solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ as a linear combination of vectors $\vec{X}_k(t)$, i.e. $\vec{x} = \sum_{k=1}^{n} c_k \vec{X}_k$. If we put the constants $c_k, k = 1, 2, dots, n$ into a vector $\vec{c}$ and make the set of fundamental solutions $\vec{X}_k(t)$ the columns of a matrix $\Phi(t)$ then we can re-write the linear combination as a simple matrix multiplication, i.e. $\vec{x} = \Phi(t)\vec{c}$.

EXAMPLE (a) Show that $\vec{x} = \Phi\vec{c}$ as defined above *is* the linear combination of the fundamental set of solutions $\vec{X}_k$ (b) $\Phi'(t) = A\Phi(t)$ and (c) $\vec{x}_h = \Phi\vec{c}$ is a solution of the homogeneous system of DEs $\vec{x}' = A\vec{x}$

DEFINITION: **fundamental matrix**

The matrix $\Phi(t)$ is called the **fundamental matrix** of the system of DEs on the interval I. The determinant of $\Phi(t)$ is the Wronkskian, $W(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n) > 0$ and therefor the fundamental matrix is non-singular and its inverse $\Phi^{-1}(t)$ must exist.

3. Variation of Parameters

Just like before, given a known solution to the homogeneous problem, we obtain a solution to the nonhomogeneousproblem by assuming it has a particular form. In this case, we let $\vec{X}_p = \Phi(t)\vec{U}(t)$ be a particular solution to the non-homogeneous DE system $\vec{x}' = A\vec{x} + \vec{f}$

By plugging this formula for $\vec{X}_p$ into this last expression, eliminating common terms and using the product rule we discover that

$$\Phi(t)\vec{U}'(t) = \vec{f}(t)$$

But we can solve this expression for $\vec{U}(t)$,

$$\vec{U}(t) = \int \Phi^{-1}(t)\vec{f}(t)dt$$

so since $\vec{X}_p = \Phi(t)\vec{U}(t)$, then

$$\vec{X}_p = \Phi(t)\int \Phi^{-1}(t)\vec{f}(t)dt$$

and the general solution to the nonhomogeneous problem can be written as

$$\vec{x} = \vec{X}_h + \vec{X}_p = \Phi\vec{c} + \int \Phi^{-1}(t)\vec{f}(t)dt$$

EXAMPLE

Let's solve the initial value problem $\vec{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\vec{x} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}, \vec{x}(0) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$