Differential Equations

Math 341 Spring 2005 — MWF 8:30 - 9:25am Fowler North 2

 — http://faculty.oxy.edu/ron/math/341

Class 21: Wednesday March 23

TITLE *Homogeneous Systems of Linear Systems of First Order DEs*
CURRENT READING Zill 8.2, Edwards & Penney Handout

Homework Set #8
Zill, Section 8.1: 4*, 5*, 11*, 18* *EXTRA CREDIT 25*
Zill, Section 8.2: 7*, 13*, 20* *EXTRA CREDIT 31, 49*

SUMMARY
We will look at techniques for solving homogeneous systems of DEs of the form $\vec{x}' = A\vec{x}$ and analyze the critical points of 2-D homogeneous linear and quasi-linear systems of DEs.

RECALL Previously we assumed that when solving an n^{th} order system of DEs we would have n associated eigenvectors $\vec{v}_k$ corresponding to the n eigenvalues λ_k and thus the solution to $\vec{x}'(t) = A(t)\vec{x}(t)$ could be written as a linear combination of vectors $\vec{X}_k$ where $\vec{X}_k = \vec{v}_k e^{\lambda_k t}$, so that $\vec{x} = \sum_{k=1}^{n} c_k \vec{v}_k e^{\lambda_k t} = c_2\vec{v}_2 e^{\lambda_2 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \ldots + c_n\vec{v}_n e^{\lambda_n t}$ where $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$ are the eigenvalues and corresponding eigenvectors of the matrix A.

1. Repeated Eigenvalues

DEFINITION: **multiplicity**
An eigenvalue λ_i which is repeated m times is said to be an **eigenvalue of multiplicity** m. When the n eigenvalues are not distinct then there may or may not be n associated eigenvectors.

CASE 1
A has an eigenvalue λ_* of multiplicity $m < n$ which has m associated eigenvectors. In this case the solution has the form $\sum_{k=1}^{m} c_k \vec{v}_k e^{\lambda_* t} + \sum_{k=m+1}^{n} c_k \vec{v}_k e^{\lambda_k t}$ where v_k are the eigenvectors associated with λ_*.

CASE 2
A has an eigenvalue λ_* of multiplicity $m < n$ which has $p < m$ associated eigenvectors. In this case the solution has the form $\sum_{k=1}^{m} c_k \vec{X}_k + \sum_{k=m+1}^{n} c_k \vec{v}_k e^{\lambda_k t}$ where $\vec{X}_m = \sum_{k=1}^{m} \vec{w}_{m-k+1} \frac{t^{k-1}}{(k-1)!} e^{\lambda_* t}$ and $\vec{w}_k$ are a **chain of length k of generalized eigenvectors** of A and $\vec{v}_k$ are regular eigenvectors of A.

DEFINITION: **generalized eigenvector**

An eigenvector $\vec{w}$ associated with λ such that $(A-\lambda I)^r\vec{w} = \vec{0}$ but $(A-\lambda I)^{r-1}\vec{w} \neq \vec{0}$ is called a **generalized eigenvector of rank r**.

DEFINITION: k **chain of generalized eigenvectors**

A chain of generalized eigenvectors of length k is a set of eigenvectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ associated with an eigenvector λ such that

$$\begin{aligned} (A-\lambda I)\vec{w}_k &= \vec{w}_{k-1} \\ (A-\lambda I)\vec{w}_{k-1} &= \vec{w}_{k-2} \\ \vdots &= \vdots \\ (A-\lambda I)\vec{w}_2 &= \vec{w}_1 \end{aligned}$$

From above, we can see that the k^{th} element in a chain of generalized eigenvectors has the property that $(A-\lambda I)^k\vec{w}_k = \vec{0}$. In practice, you'll start with using this equation to compute $\vec{w}_k$ and use the chain to compute $\vec{w}_{k-1} \ldots \vec{w}_1$.

Exercise The matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (\lambda+1)^3 = 0$ and one eigenvector $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Find the generalized eigenvectors of A.

Write down the general solution of $\dfrac{d\vec{x}}{dt} = A = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix}\vec{x}$

EXAMPLE Consider $\vec{x}' = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}\vec{x}$. Let's find the general solution.

Exercise Zill, page 345, Example 5. The matrix $\begin{bmatrix} 2 & 1 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ has a single eigenvalue $\lambda = 2$ of multiplicity 3 and defect 2. It's only eigenvector is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$

DEFINITION: **defect**

The defect d of an eigenvalue is the difference $d = m - p$ between the multiplicity m and the number p of associated eigenvectors.

(a) Find the chain of generalized eigenvectors of length 2.

(b) Write down the general form of the solution.

2. 2-D Linear and Quasi-Linear Systems of DEs

We want to analyze systems which look like $x' = f(x,y), \quad y' = g(x,y)$ or $\vec{x}' = \vec{f}(\vec{x})$. If $f(x,y)$ and $g(x,y)$ are linear (or quasi-linear, i.e. approximately linear) then we can classify the critical points of the system (where $f(x_0,y_0) = 0$ and $g(x_0,y_0) = 0$ simultaneously). For example, suppose f and g have a critical point at the origin (0,0), then

$$\begin{aligned} x' = f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y &= ax + by \\ y' = g(x,y) \approx g(0,0) + g_x(0,0)x + g_y(0,0)y &= cx + dy \end{aligned}$$

The expression on the left is a Taylor (or Maclaurin) expansion of f and g about the point (0,0). If f and g are quasi-linear than near the origin this expansion is fairly accurate. In vector notation this would be $\vec{x}' = \vec{f}(\vec{0}) + J(\vec{0})\vec{x}$ where J is the Jacobian of $\vec{f}(\vec{x})$. Thus this is now a homogeneous system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and characteristic polynomial $(a-\lambda)(d-\lambda)-bc = \lambda^2-(a+d)\lambda+ad-bc = \lambda^2-\mathrm{tr(A)}\lambda+\mathrm{det(A)} = 0$

The solutions λ_1 and λ_2 to the characteristic polynomial can be classified into a number of different cases depending on the qualities the eigenvalues possess.

GROUPWORK

Your goal is to match the case # in the left column with the description of its critical point on the right (the list now is jumbled).

CASE 1: Real λ, $\lambda_1\lambda_2 > 0$	**A** (Stable) Center
CASE 2: Real λ, $\lambda_1\lambda_2 < 0$	**B** (Stable) Spiral
CASE 3: Real λ, $\lambda_1 = \lambda_2 < 0$	**C** (Stable) Node
CASE 4: Real λ, $\lambda_1 = \lambda_2 > 0$	**D**(Unstable) Node
CASE 5: Complex λ, $\mathrm{Re}(\lambda) \neq 0$	**E** (Unstable) Saddle
CASE 6: Complex λ, $\mathrm{Re}(\lambda) = 0$	**F** (Unstable) Spiral

Below you can see what the Phase Portrait around a Center, Spiral or Node looks like.

Run the CD-Rom from Zill's textbook and select **Chapter 8: Linear Phase Portrait**. Use the slide bars to obtain different values of a, b, c and d and the different kinds of eigenvalues recorded above in the Cases. Record your results in the table below.

	a	b	c	d	λ_1	λ_2	Description
CASE #							
CASE #							
CASE #							
CASE #							
CASE #							
CASE #							

For more details, see the handout from **Edwards and Penney,** *Differential Equations*, **3rd Edition, Prentice Hall: 2004.**