Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2
 http://faculty.oxy.edu/ron/math/341

Class 20: Monday March 21

TITLE *Systems of Linear Systems of First Order DEs: Theory*
CURRENT READING Zill, 8.1 and 8.2

Homework Set #8
Zill, Section 8.1: 4*, 5*, 11*, 18* *EXTRA CREDIT 25*
Zill, Section 8.2: 7*, 13*, 20* *EXTRA CREDIT 31, 49*

SUMMARY
We will begin our study of systems of linear differential equations.

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1. Theory of Linear Systems

Consider the following system of n linear first order differential equations

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3 + \ldots + a_{1n}(t)x_n + f_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3 + \ldots + a_{2n}(t)x_n + f_2(t) \\
x_3' &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3 + \ldots + a_{3n}(t)x_n + f_3(t) \\
\vdots &= \vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + a_{n3}(t)x_3 + \ldots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

The linear system is said to be **homogeneous** if the functions $f_i(t)$ are all identically zero, otherwise the systen is called **nonhomogeneous**.

The usefulness of the above form of a linear system is that it can be written in matrix form as $\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t)$ (nonhomogeneous) and simply $\vec{x}' = A\vec{x}$ (homogeneous).

EXAMPLE Write $x' = x + 3y, \quad y' = 5x + 3y$ in matrix form and confirm that $\vec{x}_1 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$ are solutions.

Recall when we looked at n^{th} order linear differential equations we said they are very similar to systems of n first order linear differential equations.

Question How can we show that a set of solution vectors $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, ..., $\vec{x}_n$ are linearly independent solutions of a homogeneous system of linear DEs?
Answer: Use the ____!

Exercise Confirm that the given solutions in the previous example are linearly independent.

THEOREM Given $A(t)$ and $\vec{f}(t)$ are continuous functions on an interval I containing t_0 then the initial value problem $\vec{x}'(t) = A(t)\vec{x}(t)+\vec{f}(t), \quad \vec{x}(t_0) = \vec{x}_0$ possesses a unique solution on the interval I.

The unique solution $\vec{x}_h$ to a homogeneous system of linear DEs can be written as a linear combination of the fundamental set of solutions. In other words, $\vec{x}_h(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_n\vec{x}_n$. The general solution of a nonhomogeneous system can be written as a sum of the homogeneous solution and a particular solution, i.e. $\vec{x} = \vec{x}_h + \vec{x}_p$ where $\vec{x}_h' = A\vec{x}_h$ and $\vec{x}_p' = A\vec{x}_p + \vec{f}$

This solution $\vec{x}(t)$ is really a curve in space defined parametrically by the components $x_1(t), x_2(t), \ldots, x_n(t)$ known as a **trajectory**.

2. General Solution To Homogeneous Linear Systems

Consider again the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{x}$. We can write the general solution as $\vec{x} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 3 \\ 5 \end{bmatrix}e^{6t} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$. Note that both solution vectors in the fundamental set of solution $\vec{x}_1$ and $\vec{x}_2$ can be written as $\vec{v}e^{\lambda t}$.

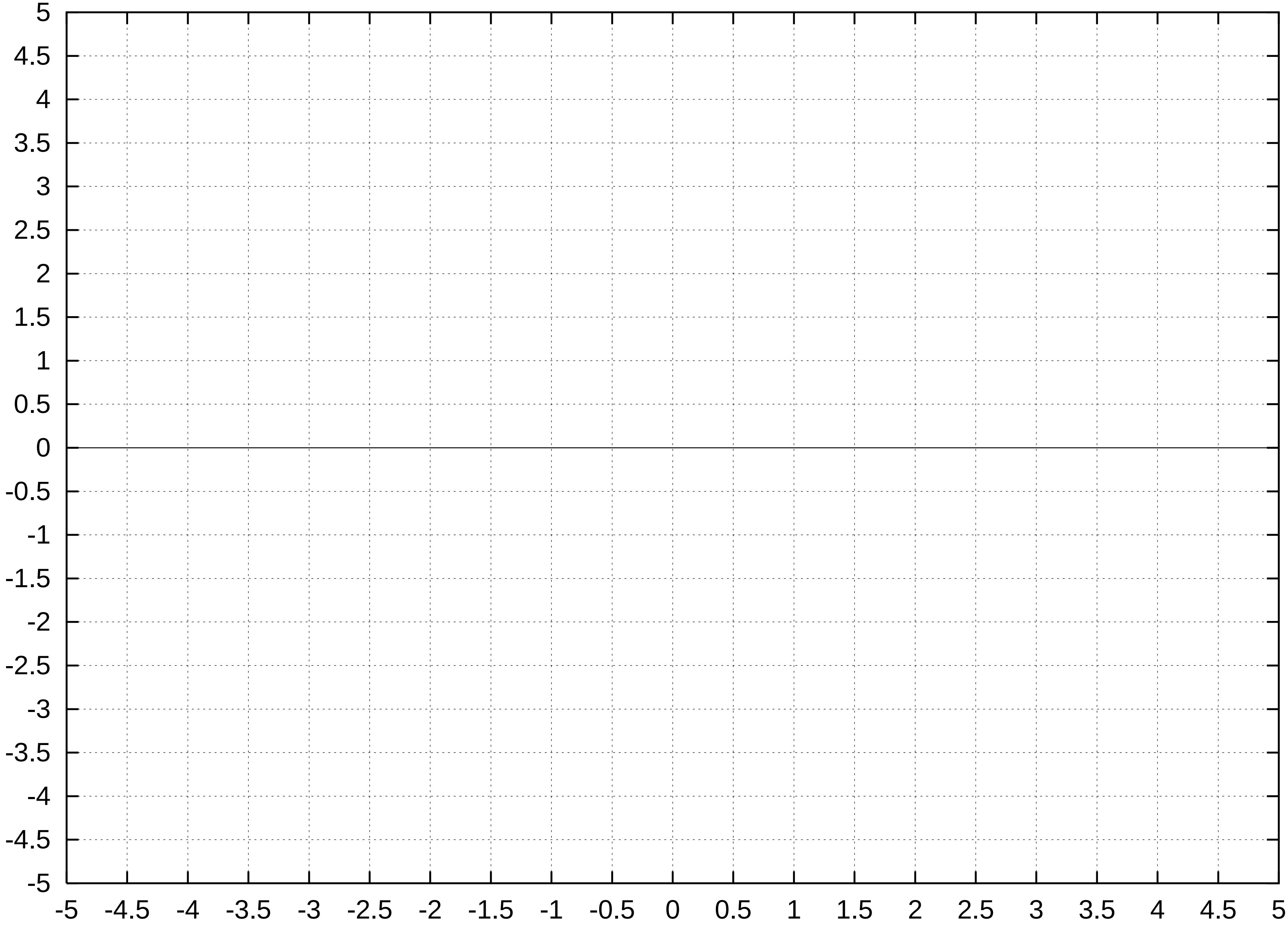
Exercise Write down the $\vec{v}$ and λ for each solution vector.

Question Do you notice anything interesting about the vector $\vec{v}$ and number λ for each solution? Any relationship to the matrix A?
Answer The vectors in question are ____.

THEOREM The general solution $\vec{x}(t)$ on the interval $(-\infty, \infty)$ to a homogeneous system of linear DEs $\vec{x}'(t) = A(t)\vec{x}(t)$ can be written as $\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} + c_3\vec{v}_3e^{\lambda_3 t} + \ldots + c_n\vec{v}_ne^{\lambda_n t}$ where $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,\ \lambda_n$ and $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3,\ \ldots,\ \vec{v}_n$ are the eigenvalues and corresponding eigenvectors of the matrix A.

Exercise **Zill, page 339, Example 1**. Solve $\dfrac{dx}{dt} = 2x + 3y, \quad \dfrac{dy}{dt} = 2x + y.$

EXAMPLE Let's sketch the phase portrait of this system.
The solution functions are $x(t) = c_1e^{-t} + 3c_2e^{4t}, \quad y(t) = -c_1e^{-t} + 2c_2e^{4t}.$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