Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2
 http://faculty.oxy.edu/ron/math/341

Class 17: Monday February 28

TITLE *Solving Nonhomogeneous DEs with Constant Coefficients*
CURRENT READING Zill, 4.4 and 4.5

SUMMARY
We will investigate techniques for finding solutions of solving nonhomogeneous DEs with constant coefficients: **the method of undetermined coefficients**.

1. Method of Undetermined Coefficients

We are considering Linear Constant Coefficient n^{th} Order DEs which have the form $Lu = g(x)$ where $g(x)$ is either a constant function, a polynomial function, a (simple) exponential function, sine or cosine or some finite sum or product of these functions.

Exercise Consider the following functions $g(x)$. Which of these will the Method of Undetermined Coefficients solve $Lu = g$?

1. $g(x) = \ln(x)$
2. $g(x) = (2x^2 - 3x + 4)\sin(3x)$
3. $g(x) = e^{x^2}\cos(3x)$
4. $g(x) = e^x\cos(3x)$
5. $g(x) = x^2e^x\cos(3x)$
6. $g(x) = 7$
7. $g(x) = 2/x$
8. $g(x) = \tan(x)$
9. $g(x) = e^{-7x}(x + 4)$
10. $g(x) = (x + 4)^7$

EXAMPLE Let's use the Method of Undetermined Coefficients to solve
$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$

2. Formalizing The Method

First, find the fundamental set of solutions $y_h(x)$ to the homogeneous analogue $Ly = 0$ to the given problem $Ly = g$

Second, examine the source function $g(x)$ and guess a corresponding particular solution $y_p(x)$.

Third, substitute your guess for $y(x)$ into $Ly = g$ and group terms in order to find the undetermined coefficients.

Form of $g(x)$	Choice of $y_p(x)$
42 (Any $C \neq 0$)	A
$3x + 5$	$Ax + B$
$2x^2 - 4x + 4$	$Ax^2 + Bx + C$
$x^3 - 1$	$Ax^3 + Bx^2 + Cx + D$
x^n	$\sum_{k=0}^{n} c_k x^k$
$\sin(4x)$	$A\sin(4x) + B\cos(4x)$
$\cos(4x)$	$A\sin(4x) + B\cos(4x)$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{5x}\sin(2x)$	$Ae^{5x}\cos(2x) + Be^{5x}\cos(2x)$
$x^2\sin(2x)$	$(Ax^2 + Bx + C)\cos(2x) + (Dx^2 + Ex + F)e^{5x}\cos(2x)$
$xe^{5x}\sin(2x)$	$(Ax + B)e^{5x}\cos(2x) + (Cx + D)e^{5x}\cos(2x)$

Rules for Methods of Undetermined Coefficients (*Zill*)
Rule 1 The form of $y_p(x)$ is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$

Rule 2 If any part of $y_p(x)$ contains terms that duplicate terms in y_h then that part of y_p must be multiplied by x^n, where n is the smallest positive integer that eleiminates that duplication.

Exercise Find the solution of $y'' - 2y' + y = e^x$.

3. Higher Order Examples

EXAMPLE Solve $y''' + y'' = e^x \cos(x)$

Exercise Determine the particular solution of $y^{(4)} + y''' = 1 - x^2 e^{-x}$

4. Annihilator Approach

DEFINITION: **annihilator**

A linear operator L is said to be an **annihilator** or **annihilator operator** for a function $f(x)$ if when L is applied to f zero results; in other words $L[f] = 0$.

EXAMPLE What are the annihilator operators for the following functions:

(a) $f(x) = x^n$

(b) $f(x) = e^{mx}$

(c) $f(x) = x^n e^{mx}$

(d) $f(x) = \cos(\beta x)$

(e) $f(x) = e^{mx}\cos(\beta x)$

(f) $f(x) = x^n e^{mx}\cos(\beta x)$

EXAMPLE Let's use the annihilator approach to find the particular solution of

(a) $y'' - 2y' + y = e^x$ (b) $y''' + y'' = e^x\cos(x)$ and (c) $y^{(4)} + y''' = 1 - x^2e^{-x}$

Exercise Use the annihilator approach and the method of undetermined coefficients to determine the particular solution for $y'' - 2y' + y = 10e^{-2x}\cos(x)$