Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2
 `http://faculty.oxy.edu/ron/math/341`

Class 16: **Friday February 25**

TITLE *Homogeneous Linear ODEs with Constant Coefficients*
CURRENT READING Zill, 4.3

SUMMARY
We will investigate techniques for finding solutions of homogeneous linear ODEs with constant coefficients.

Homework Set #6
Zill, Section 4.2: 2*, 11*, 19*
Zill, Section 4.3: 6*, 16*, 23*, 33* *EXTRA CREDIT 43,44,45,46,47,48*
Zill, Section 4.6: 5*, 16*, 19* *EXTRA CREDIT 30*

We will begin by looking at solution techniques for solving the linear 2^{nd} order DE $ay'' + by' + cy = 0$ where a, b and c are constants.

1. Auxiliary Equation

Let's guess that the solution to Equation is $y = e^{mx}$. (In Physics, we would say "Let's make an *ansatz* of $y = e^{mx}$). By guessing $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$ we obtain $(am^2 + bm + c)e^{mx} = 0$ from which we know either $e^{mx} = 0$ or $am^2 + bm + c = 0$. The latter is known as the **auxiliary equation**.

Clearly there are three distinct types of solutions to this equation, depending on the values of a, b and c.
Case I: Two distinct real roots (when $b^2 - 4ac > 0$)
Case II: Two indistinct real roots (when $b^2 - 4ac = 0$)
Case III: Two distinct complex roots (when $b^2 - 4ac < 0$)

Case I: Two Real Roots
If there are two real roots, m_1 and m_2 then the fundamental set of solutions to the homogeneous linear DE is simply $y = c_1e^{m_1x} + c_2e^{m_2x}$ where the auxiliary equation $am^2 + bm + c = 0$ can be factored as $(m - m_1)(m - m_2) = 0$

Case II: One Repeated Real Root
If there is only one real root then we know we have one solution $y_1(x) = e^{m_1x}$ and the auxiliary equation can be factored as $(m - m_1)^2 = 0$. We can get a second solution by using the method of reduction of order where $P(x) = b/a$ and $Q(x) = c/a$ to show that the fundamental set of solutions is $y = c_1e^{m_1x} + c_2xe^{m_1x}$.

Exercise Show that when $ay'' + by' + cy = 0$ has one repeated root $m = m_1$ in the auxiliary equation and one solution $y = e^{m_1x}$ then another solution is $y_2(x) = xe^{m_1x}$.

Case III: Two Complex Roots
If there are two complex roots, then they are complex conjugate pairs $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and β are real numbers and $i^2 = -1$.
The fundamental set of solutions in this case will be $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$
RECALL $e^{i\theta} = \cos\theta + i\sin\theta$

EXAMPLE Let's use these results to solve the equations
(a) $2y'' - 5y' - 3y = 0$
(b) $y'' - 10y' + 25y = 0$
(c) $y'' + 4y' + 7y = 0$

2. Higher Order Constant Coefficient Linear DEs

Things get more complicated when the order of the equation goes up, but the basic idea is the same. Consider the general n^{th} order linear constant coefficient DE, the auxiliary equation will have the form

$$a_n m^n + a_{n-1} m^{n-1} + a_{n-2} m^{n-2} + \ldots + a_2 m^2 + a_1 m + a_0 = 0$$

If all the n roots are real and distinct, then the fundamental solution will be inspired by case I:
$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If there are k repeated real roots (and $n-k$ distinct real roots) the fundamental solution will be inspired by Case II:
$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \ldots + c_k x^{k-1} e^{m_1 x} + d_1 e^{m_2 x} + d_2 e^{m_3 x} + \ldots d_{n-k} e^{m_{n-k} x}$

If there are k repeated complex roots (and $n-k$ distinct complex roots) the fundamental solution will be inspired by Case III:
$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 x e^{\alpha x}\cos(\beta x) + c_3 x^2 e^{\alpha x}\cos(\beta x) + \ldots + c_k x^{k-1} e^{\alpha x}\cos(\beta x) + d_1 e^{\alpha x}\sin(\beta x) + d_2 x e^{\alpha x}\sin(\beta x) + d_3 x^2 e^{\alpha x}\sin(\beta x) + \ldots + d_k x^{k-1} e^{\alpha x}\cos(\beta x) + p_1 e^{\alpha_2 x}\cos(\beta_2 x) + q_1 e^{\alpha_2 x}\sin(\beta_2 x) + p_2 e^{\alpha_3 x}\cos(\beta_3 x) + q_2 e^{\alpha_3 x}\sin(\beta_3 x) + \ldots p_{n-k} e^{\alpha_2 x}\cos(\beta_{n-k} x) + q_{n-k} e^{\alpha_{n-k} x}\sin(\beta_{n-k} x)$

Regardless, there are always n unknown functions in the fundamental set of solutions of an n^{th} order linear constant coefficient DE.

Exercise Write down the fundamental solution of 7^{th} order differential equation which has the auxiliary equation $(m^2 + 2m + 4)^2 (m-1)^2 (m+4) = 0$.