Differential Equations

Math 341 Spring 2005 — MWF 8:30 - 9:25am Fowler North 2

 — http://faculty.oxy.edu/ron/math/341

Class 14: **Friday February 18**

TITLE *Solving Linear ODEs: Reduction of Order and Variation of Parameters*
CURRENT READING Zill, 4.2 and 4.6

Homework Set #6
Zill, Section 4.2: 2*, 11*, 19*
Zill, Section 4.3: 6*, 16*, 23*, 33* *EXTRA CREDIT 43,44,45,46,47,48*
Zill, Section 4.6: 5*, 16*, 19* *EXTRA CREDIT 30*

SUMMARY
We will investigate some techniques for finding solutions of second order linear DEs.

1. The Method of Reduction of Order

Consider the general homogeneous $\mathbf{2^{nd}}$ **order linear DE** can be written

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad (1)$$

If we know a solution $y_1(x)$ which solves the above equation, we can find another by using **the method of reduction of order**. This will involve assuming the second solution has the form $y_2(x) = u(x)y_1(x)$ and showing that we can find $u(x)$ by solving a linear first order DE (using the ________________ method).

Assuming that $a_2(x)$ is not zero anywhere in the interval I that your given solution $y_1(x)$ is defined one can divide Equation 1 by $a_2(x)$ to obtain the standard form:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad (2)$$

By making the substitution $y = u(x)y_1(x)$ where $y_1(x)$ solves 2 above, show that u satisfies the equation

$$y_1u'' + (2y_1' + Py_1)u' = 0$$

By letting $w = u'$ we can obtain a linear DE for w

$$w' + \left(2\frac{y_1'}{y_1} + P\right)w = 0$$

We can show that this equation has the solution $w(x) = c_1\dfrac{e^{-\int P(x)dx}}{(y_1(x))^2}$ and since $w = u'$ $u(x) = c_1\displaystyle\int \frac{e^{-\int P(x)\ dx}}{(y_1(x))^2}dx + c_2$ which leads us to the form of the second solution $y_2(x)$ to Equation (2) given by:

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)\ dx}}{(y_1(x))^2}dx$$

(Remember: $y_2(x) = u(x)y_1(x)$)

Exercise 1 Show that $y_2(x)$ above satisfies the equation $y'' + P(x)y' + Q(x)y = 0$.

EXAMPLE **Zill, Example 2, page 141.** Given that $y_1(x) = x^2$ is a solution to $x^2y'' - 3xy' + 4y = 0$ find the general solution of this differential equation on $(0, \infty)$.

2. The Method of Variation of Parameters

Consider the nonhomogeneous version of Equation (2),

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = f(x) \tag{3}$$

If we have two solutions to the homogeneous version of the DE, called $y_1(x)$ and $y_2(x)$, then we can attempt to find two solutions to Equation (3) by assuming the particular solution will have the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $y_h = c_1y_1(x) + c_2y_2(x)$ is the solution to the corresponding homogeneous problem.

By carefully differentiating, we can obtain expressions for $y_p'(x)$ and $y_p''(x)$ (which will contain u_1, u_2, y_1 and y_2 and their derivatives)

If we combine these expressions and substitute into Equation (3) we will obtain:

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x)$$

This equation will be identically solved if

$$\begin{aligned} y_1u_1' + y_2u_2 &= 0 \\ y_1'u_1' + y_2'u_2' &= f(x) \end{aligned}$$

Questions

What are the *unknowns* in these equations? ____________________ and ____________________

What are the *knowns* in these equations? ____________________ and ____________________

We can solve these equations using **Cramer's Rule**!

EXAMPLE Let's show that we can use Cramer's Rule to obtain the formulas:

$$u_1' = \frac{-f(x)y_2(x)}{y_1y_2' - y_1'y_2} \quad \text{and} \quad u_2' = \frac{f(x)y_1(x)}{y_1y_2' - y_1'y_2}$$

(Note the denominator should look familiar: It is the ____________________ of $y_1(x)$ and $y_2(x)$.)

Exercise 2 Consider $y''-4y'+4y=0$. Show that $y_1(x)=e^{2x}$ and $y_2=xe^{2x}$ are solutions of the homogeneous DE.U se this information to obtain the solution of $y''-4y'+4y=(x+1)e^{2x}$.

Next we shall concentrate on methods to solve homogeneous linear DEs where we are not given any prior information (although we are going to generally require that the functions $P(x)$ and $Q(x)$ be pretty simple, usually constants or polynomials. This will be covered in Sections 4.3, 4.4/4.5 and 4.7 in *Zill*.