Differential Equations

Math 341 Spring 2005 — MWF 8:30 - 9:25am Fowler North 2

 — http://faculty.oxy.edu/ron/math/341

Class 13: Wednesday February 16

TITLE *Analyzing Linear ODEs of Any Order*
CURRENT READING Zill, 3.3

Homework Set #5
Zill, Section 3.3: 1*, 15* *EXTRA CREDIT Page 121, #8; Page 122 #18*
Zill, Section 4.1: 5*, 7*, 8*, 12* *EXTRA CREDIT 36, 37*

SUMMARY
We will examine properties of n-th order linear differential equations and make some interesting connections to linear systems of algebraic equations.

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1. Existence and Uniqueness of Solutions

The $\mathbf{n}^{th}$ **order initial value problem** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

with $y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

THEOREM
Let $a_0(x), a_1(x), a_2(x), a_3(x), \ldots, a_n(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in the interval I. If $x = x_0$ is any point in I, then a unique solution $y(x)$ exists on the interval I.

This is the n^{th} order version of the existence and uniqueness theorem we learned for first-order DEs earlier.

2. Differential Operators

The act of differentiating a function $y(x)$ with respect to its independent variable, often denoted $\dfrac{dy}{dx}$ can also just simply be denoted Dy. The symbol D is called a **differential operator**. Higher-order derivatives can be represented by multiple application of the operator, so $\dfrac{d^3y}{dx^3} = D(D(Dy))) = D^3 y$

Differential operators can be combined into polynomial expressions which are also differential operators. In fact, the n^{th}-order differential operator is defined as

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + a_{n-2}(x)D^{n-2} + \ldots + a_2(x)D^2 + a_1(x)D + a_0(x)$$

so that an n^{th}-order linear differential equation can be written simply as
$Ly = g(x)$ (non-homogeneous version) or $Ly = 0$ (homogenenous version).

Linearity
Thanks to the rules of differentiation, the n^{th} order differential operator L is a **linear operator**.
Specifically, $L[\alpha f(x)+\beta g(x)] = \alpha L[f(x)] + \beta L[g(x)]$

EXAMPLE Lets show that $(D^2+9)[\cos(3x)] = 0$

3. Homogeneous Linear n^{th} order DEs

A consequence of L being a linear operator is that any solution of an n^{th} order linear DE can be written as a **superposition** or **linear combination** of other solutions to the DE.

THEOREM
Let $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ be solutions of an n^{th}-order linear homogeneous DE on an interval I. The linear combination $y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is also a solution of the DE, on the same interval, where c_i for $i = 1, \ldots, n$ are arbitrary constants.

Corollary
(1) Any constant multiple $y = cy_h(x)$ of a given solution $y_h(x)$ is **also** a solution of a homogeneous linear DE $Ly = 0$
(2) A homogeneous linear DE $Ly = 0$ always posseses the trivial solution $y = 0$
Exercise Prove these two corollaries.

RECALL A set of vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ is said to be **linearly dependent** if there exist constants c_1, c_2, …, c_n not all zero such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. The vectors are said to be **linearly independent** if the only solution to the same equation is that $c_1 = c_2 = c_3 = \ldots c_n = 0$.

Interestingly, we can apply the identical idea to functions:

DEFINITION: **linear dependence**
A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants c_1, c_2, …, c_n not all zero such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) + \ldots + c_nf_n(x) = 0$ for every x in the interval I. If the set of functions is not linearly dependent, they are said to be linearly independent.

Exercise Are the functions e^x and e^{-x} linearly dependent? What about e^x, e^{-x} and $\cosh(x)$? How do we find solve this problem?

DEFINITION: Wronskian

The **Wronskian** of a set of n functions is denoted by the symbol $W(f_1, f_2, \ldots, f_n)$ and is defined as the determinant of a matrix composed of columns which have the functions and their first $(n-1)$ derivatives, in other words:

$$W(f_1, f_2, f_3, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

EXAMPLE Let's compute the Wronskian of e^x, e^{-x} and $\cosh(x)$

THEOREM

Let $y_1(x), y_2(x), y_3(x), \ldots, y_n(x)$ be solutions of an n^{th}-order linear homogeneous DE on an interval I. The set of solution is **linearly independent** on I if and only if $W(y_1, y_2, y_3, \ldots, y_n) \neq 0$ for every x in the interval I.

Question: Do you see any connections between the above theorem and the criteria for when solutions of the linear system of equations $A\vec{x} = \vec{b}$ exist?

THEOREM

The n^{th}-order linear homogeneous DE possesses linearly independent solutions $y_1(x), y_2(x), y_3(x)$, $\ldots, y_n(x)$ on an interval I known as the **fundamental set of solutions**. The **general solution** of the n^{th} order linear homogeneous DE is $y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ where c_1, c_2, $\ldots$, c_n are arbitrary constants.

4. Nonhomogeneous Linear n^{th} order DEs

Now that we have the general form of the solution to the homogeneous linear n^{th} order DE we can consider the solution of non-homogeneous linear DEs. Given a particular solution y_p which solves the nonhomogenous linear DE $Ly_p = g(x)$ and a fundamental set of solutions to the homogeneous linear DE $Ly_h = 0$ the general solution of the nonhomogeneous DE can be written as $y = y_p + y_h$, where $y_h = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ and $y_i(x)$ for $i = 1, \ldots, n$ are all solutions of $Ly = 0$.

$\boxed{\text{EXAMPLE}}$ **Zill, Example 9, page 134.** Show that $y_1 = e^x$, $y_2 = e^{2x}$ and $y_3 = e^{3x}$ satisfy the homogeneous DE $y''' - 6y'' + 11y' - 6y = 0$. Also show that these three functions form a fundamental set of solutions to the homogeneous DE.

Zill, Example 10, page 135. Show that the function $y_p(x) = -\frac{11}{12} - \frac{1}{2}x$ is a particular solution to the nonhomogeneous DE $y''' - 6y'' + 11y' - 6y = 3x$ and write down the general solution of the nonhomogeneous DE.