Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2
 http://faculty.oxy.edu/ron/math/341

Class 9: Monday February 7

TITLE *Analyzing Linear and Nonlinear Autonomous First-Order DE Models*
CURRENT READING Zill, 3.1 and 3.2

Homework Set #4
Zill, Section 2.4: 5, 10*, 11*, 19, 27*, 30, 32, 38*, *EXTRA CREDIT 42, 44*
Zill, Section 3.1: 1*, 10, 17*, 19*, *EXTRA CREDIT 37*
Zill, Section 3.2: 5, 13*, 19*, *EXTRA CREDIT 22*

SUMMARY
We will revisit some standard first-order, autonomous DE models and begin our analysis of modified versions of them.

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1. Linear Autonomous DE

RECALL The standard exponential or Malthusian Population Model is $P' = kP, P(0) = P_0$ with solution $P(t) = P_0e^{kt}$

However this model ignores many other effects that would increase or decrease the population, like immigration (constant input) or emigration (constant decrease). The most general form of a linear autonomous DE is $P' = aP + b, \quad P(0) = P_0$

Questions we want to analyze are:

- What impact do various parameter values have on the nature of the solution?
- What different kind of systems are we modelling with different parameter values?

Exercise Find the particular solution for the general form of the linear autonomous DE.

Analysis
Let's restrict ourselves to phenomena where $P \geq 0$ for all $t \geq 0$. What long term (equilibrium) behavior is preidcted by our model? How does it depend on a and b?

2. Quadratic Autonomous DE

RECALL The standard Verhulst or logistic population model is

$P' = kP(1 - P/M), P(0) = P_0$ with solution $P(t) = \dfrac{P_0 M e^{kt}}{(M - P_0) + P_0 e^{kt}}$

The most general form of a quadratic DE would be $P' = aP^2 + bP + c, \qquad P(0) = P_0$

The logistic model corresponds to $a < 0$, $b > 0$ and $c = 0$. Typically $|b| >> |a|$.
In this model, the parameter a corresponds to an "inhibition" or competition term while b is similar to the rate constant from the exponential growth model and c represents a constant increase/decrease in population.

Modifications of the Logistic Model

How would the Logistic DE change if immigration into the country was proportional to the current population?

$$P' = kP(1 - P/M) + cP, \qquad P(0) = P_0$$

How would the Logistic DE change if the population controlled immigration so that when the population was large the rate of immigration was very low, but when the population is small, the rate of immigration is large.

$$P' = kP(1 - P/M) + ce^{-dP}, \qquad P(0) = P_0$$

Gompertz Equation

The following modification of the logistic equation is often used to measure the growth of tumors:

$$P' = kP(1 - c\log P), \qquad P(0) = P_0$$

Exercise Show that the Gompertz equation has equilibrium points at 0 and $e^{1/c}$ and confirm the exact form of the solution is $e^{\frac{1}{c}\left(1-(1-c\ln P_0)e^{-ckt}\right)}$.