Differential Equations

Math 341 Spring 2005 | MWF 8:30 - 9:25am Fowler North 2

 | `http://faculty.oxy.edu/ron/math/341`

Class 8: **Friday February 4**

TITLE *Substitution Techniques for Solving First-Order DEs*
CURRENT READING Zill, 2.5

SUMMARY
We will be introduced to another technique for finding the solution of a DE when previously discussed methods (separation of variables, integrating factors, exact differentials) don't seem to apply. The idea is to transform the given DE into another DE using a **substitution**.

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1. Substitution Examples
Consider the differential equation $\dfrac{dy}{dx} = f(x, y)$. If we assume that the solution will have the form $y = g(x, u)$ where $u = u(x)$ use the Chain Rule to show that the transformed differential equation will be $\dfrac{du}{dx} = F(x, u)$ where F depends on f and partial derivatives of g.

2. Substitution Example
Suppose $\dfrac{dy}{dx} = F(Ax + By + C)$. If we make the substitution $u(x) = Ax + By(x) + C$ then this differential equation can be transformed by $\dfrac{du}{dx} = A + B\dfrac{dy}{dx}$ into $\dfrac{du}{dx} = A + BF(u)$. What technique can we use to obtain an expression for $u(x)$ and thus $y(x)$?

EXERCISE **Zill, page 77, Example 3.** Solve $y' = (-2x + y)^2, \quad y(0) = 0$

3. Bernouilli Differential Equations

A differential equation which has the form $y' + P(x)y = Q(x)y^n$ where n is an real number is called a **Bernouilli Equation**. For $n = 0$ and $n = 1$ we already know how to solve the DE. **How?** For $n \neq 0$ and $n \neq 1$ the substitution $u(x) = y^{1-n}$ reduces a Bernouilli equation to a linear DE.

$\boxed{\textbf{EXERCISE 2}}$ Show that a Bernouilli Equation $y' + P(x)y = Q(x)y^n$ can be solved for any value of n

4. Riccati Differential Equations

A nonlinear differential equation which has the form $y' = A(x)y^2 + B(x)y + C(x)$ is called a **Riccati Equation**. If one knows that $y = y_1(x)$ is a solution of the Riccati Equation, by using the substitution $u(x) = \dfrac{1}{y - y_1(x)}$ one can transform it into

$\dfrac{du}{dx} + (B(x) + 2A(x)y_1(x))v = -A(x)$

5. Clairault's Differential Equations

A differential equation which has the form $y = xy' + g(y')$ is called a **Clairault Equation**. It's solution has the form $y(x) = Cx + g(C)$ where C is an unknown constant.

6. Homogeneous Differential Equations

DEFINITION: **homogeneous functions**

A function which has the property that $f(tx, ty) = t^\alpha f(x, y)$ is said to be a **homogeneous function of degree** α. For example, $f(x, y) = x^3 + y^3$ is homogeneous, while $f(x, y) = x^3 + y^3 + 1$ is not homogeneous. NOTE: this use of homogeneous has nothing to do with the more common use of homogeneous, meaning a solution which solves the version of an equation where the right hand side is zero.

Sadly, the textbook describes a first order DE which has the following form

$$M(x, y)\ dx + N(x, y)\ dy = 0$$

where $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree as a **homogeneous first-order differential equation**.

In other words, if $M(x, y) = x^\alpha M(\frac{y}{x})$ and $N(x, y) = x^\alpha N(\frac{y}{x})$ then $u = y/x$ is a substitution which will transform the homogeneous differential equation into separable differential equation.

Similarly, if $M(x, y) = y^\alpha M(\frac{x}{y})$ and $N(x, y) = y^\alpha N(\frac{x}{y})$ then $v = x/y$ is a substitution which will transform the homogeneous differential equation into separable differential equation.

Proof Let's show how the $u = y/x$ and $v = x/y$ substitutions "work."

EXAMPLE Solve $(4x^2 + 3y^2)\ dx - 2xy\ dy = 0$

$\boxed{\textbf{EXERCISE 3}}$ Consider the differential equation

$$\frac{dy}{dx} = x^3(y - x)^2 + \frac{y}{x}$$

(a) Show that $y = x$ is a solution to the differential equation.

(b) Find all the other solutions to this differential equation.