Differential Equations

Math 341 Spring 2005 | MWF 8:30 - 9:25am Fowler North 4

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Class 3: Monday January 24

TITLE Mathematical Models: Using Differential Equations To Describe Nature
CURRENT READING Zill, 1.2 and 1.3

Homework Set #2
Section 1.2 # 1, 2*, 3, 4*, 5, 6*, 15, 17, 20*, *EXTRA CREDIT: 31, 32*
Section 1.3 # 1, 2, 3*, 5*, 8*, *EXTRA CREDIT: 13, 14*
Section 2.1:# 1, 2*, 3, 4*, 5, 6*, 8*, 12, 17*, 21, 22 *EXTRA CREDIT 19, 29, 32, 37*
EXTRA CREDIT HW Set: CHAPTER 1 REVIEW: # 5, 7, 8, 9, 10, 15, 16, 17, 21, 22

SUMMARY
We'll cement our understanding of the Existence and Uniqueness Theorem with some more examples. Then we will be introduced to some IVPs which represent mathematical models of real-world processes.

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1. Understanding the Existence and Uniqueness Theorem

EXAMPLE

Consider the initial value problem $x\dfrac{dy}{dx} - 2y = 0, \quad y(a) = b$ Discuss the dependence of the existence of unique solutions upon the location of the initial condition (value of a and b).

STEP 1: Identify $f(x, y)$ and compute $f_y(x, y)$.

STEP 2: Find the set of points in the plane for which $f(x, y)$ and $f_y(x, y)$ are both continuous.

STEP 3: See if the initial condition is in the set you found in STEP 2. If YES, the IVP possesses a unique solution through that initial condition. If NO, there may be NO solution, or multiple solutions which go through that point.

GroupWork

Consider four cases for the location of the initial condition (a, b) in the above IVP and describe the nature of the solutions for each case.

Case I $a = 0, b = 0$

Case II $a = 0, b \neq 0$

Case III $a \neq 0, b = 0$

Case IV $a \neq 0, b \neq 0$

2. IVPs as Mathematical Models

DEFINITION: **mathematical model**

A **mathematical model** is a mathematical description of a system or phenomenon. Many physical systems often involve time (the variable t) so that the mathematical description of the model involves differential equations and the solution of the model produces a state of the system at certain points in time: the past, present or future. As Zill points out, "A single differential equation can serve as a mathematical model for many different phenomena" (21).

In Section 1.3 of the text, Zill introduces a number of differential equations which serve as mathematical models of various physical systems and phenomena.

Often mathematical models begin with a relationship between the rate of change of a quantity and some other quantity.

Malthusian Population Model

A mathematical model of human population growth was introduced by an English economist named Thomas Malthus in 1798. The main assumption is that the rate of growth of the population is proportional to the current population. Mathematically,

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$$

This model actually pretty accurately reflected the population growth of the United States in its early stages of development.

Logistic Population Model of Verhulst

In 1846 a Belgian mathematician named Pierre Verhulst developed another model of human population growth. Its main assumption is that the relative rate of growth is proportional to the excess population an environment can hold. Instead of the relative rate of growth being constant, it decreases with the growth of the population itself. This model is know known as the **logistic model**. Mathematically,

$$\frac{dP}{dt}\frac{1}{P} \propto 1 - \frac{P}{M} \Rightarrow \frac{dP}{dt} = kP(1 - P/M)$$

Differential equations are used to map all sorts of physical phenomena, from chemical reactions, disease progression, motions of objects, electronic circuits, et cetera. Most mathematical models of real-world situations do not have analytical solutions. For example, in 2001 I published a paper with a differential equation model of how movies make money (Edwards and Buckmire, "A differential equation model of north american cinematic box-office dynamics," *IMA Journal of Management Mathematics*, **12**(1)). The model is represented by an IVP:

$$\begin{aligned}
\frac{dG}{dt} &= SA, \qquad G(0) = 0 \\
\frac{dS}{dt} &= -(S - A) \qquad S(0) = S_0 \\
\frac{dA}{dt} &= -\alpha\left(\frac{S}{S+\gamma} + \beta G\right) A, \qquad A(0) = A_0
\end{aligned}$$