Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 4

 `http://faculty.oxy.edu/ron/math/341`

Class 2: **Friday January 21**

TITLE Initial Value Problems: Definition, Existence, Uniqueness
CURRENT READING Zill, 1.1 and 1.2

Homework Assignments for Chapter 1
Section 1.1 # 1, 2, 3, 4, 8, 9, 11, 12, 15, 17, 21, 22, 27, 28, 29, 30, 31, *EXTRA CREDIT: 41, 42, 43, 44, 51* (**Hand in # 2, 4, 8, 12, 17, 22, 28, 30 ON FRI JAN 21**)
Section 1.2 # 1, 2, 3, 4, 5, 6, 15, 17, 20, *EXTRA CREDIT: 31, 32* (**Hand in # 2, 4, 6, 20 ON FRI JAN 28**)
Section 1.3 # 1, 2, 3, 5, 8, *EXTRA CREDIT: 13, 14* (**Hand in # 3, 5, 8** ON FRI JAN 28)
CHAPTER 1 REVIEW: **Hand in # 5, 7, 8, 9, 10, 15, 16, 17, 21, 22 on FRI JAN 28**

SUMMARY
In today's class we shall consider initial value problems, and be introduced to the most important theorem(s) in the study of differential equations: The Existence and Uniqueness Theorems.

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1. Initial Value Problems

DEFINITION: **initial value problem**

An **initial value problem** or IVP is a problem which consists of an n-th order ordinary differential equation combined with n initial conditions defined at a point x_0 found in the interval of definition I.

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{(n-1)}) \qquad \text{differential equation}$$

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \ldots, y^{(n)}(x_0) = y_n \qquad \text{initial conditions}$$

where $y_0, y_1, y_2, \ldots, y_n$ are known constants.

For example, a first-order IVP looks like $y' = f(x, y), \quad y(x_0) = y_0$ and
a second-order IVP looks like $y'' = f(x, y, y'), \quad y(x_0) = y_0$ and $y'(x_0) = y_1$.

EXERCISE

Consider the following IVPs: $y' = y, \quad y(0) = 3$ and $y' = y, \quad y(1) = -2$. Find the one-parameter family of solutions to the ODE along with its interval of definition and then sketch the solutions to the given initial value problems.

EXAMPLE

We can show that the one-parameter family of solutions to the ODE $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$. If we include the initial condition $y(0) = -1$ we can show that the corresponding value of $c = -1$ and thus the particular solution is $y(x) = \dfrac{1}{x^2 - 1}$.

(a) What is the domain of definition of the function $y(x) = \dfrac{1}{x^2 - 1}$?

(b) What is the interval of definition of the solution of the ODE? (i.e. on what sets if the function $y(x)$ defined and differentiable?)

(c) What is the interval of definition of the solution of the initial value problem? (i.e. which set contains the initial condition and the function $y(x)$ is defined and differentiable at all points?)

2. Existence and Uniqueness

The main questions we would like to be able to answer when analyzing IVPs are: 1) **Existence** Does the differential equation possess solutions which pass through the given initial condition? and 2) **Uniqueness** If such a solution does exist, can we be certain that it is the only one? Luckily, there's a theorem that answers these questions for us.

THEOREM: **Existence of a Unique Solution**

Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b, c \leq y \leq d$ that contains the point (x_0, y_0) in its interior. IF $f(x, y)$ and $\partial f/\partial y$ are continuous on R, THEN there exists some interval I_0 defined as $x_0 - h < x < x_0 + h$ for $h > 0$ contained in $a \leq x \leq b$ and a unique function $y(x)$ defined on I_0 that is a solution of the initial value problem.

EXAMPLE

Recall that the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

has at least two solutions since the trivial solution $y(x) = 0$ and the solution $y(x) = \frac{1}{16}x^4$ both satisfy the IVP. **Verify this!**

Using the Existence and Uniqueness Theorem, we look at the functions $f(x, y) = x\sqrt{y}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{y}}$. At the origin $(0, 0)$ what can we say about $f(x, y)$ and $f_y(x, y)$?

What can we say about $f(x, y)$ and $f_y(x, y)$ at $(1, 2)$? What does this imply about existence and uniqueness of the corresponding IVP $y' = xy^{1/2}, y(1) = 2$?