Test 1: Differential Equations

Math 341 Wednesday March 9, 2005

Name: ______________________

Directions: Read *ALL* four (4) problems first before answering any of them. This is a one hour, closed notes, closed book, test. This test has 7 pages. You must show all relevant work to support your answers. Use complete English sentences and indicate your final answer from your "scratch work."

No.	Score	Maximum
1		25
2		25
3		20
4		30
BONUS		10
Total		100

1. *[25 points total.]* **Families of Solutions, Existence and Uniqueness.**
Consider $\frac{dy}{dx} = 2xy^2, y(0) = \frac{1}{c^2}$ with $c > 0$.
(a) *[5 points].* For what values of c do unique solutions to the given initial value problem exist? **EXPLAIN YOUR ANSWER.**

(b) *[10 points].* Show that the one-parameter family of curves $y(x) = \frac{1}{c^2 - x^2}$ are solutions to the given initial value problem.

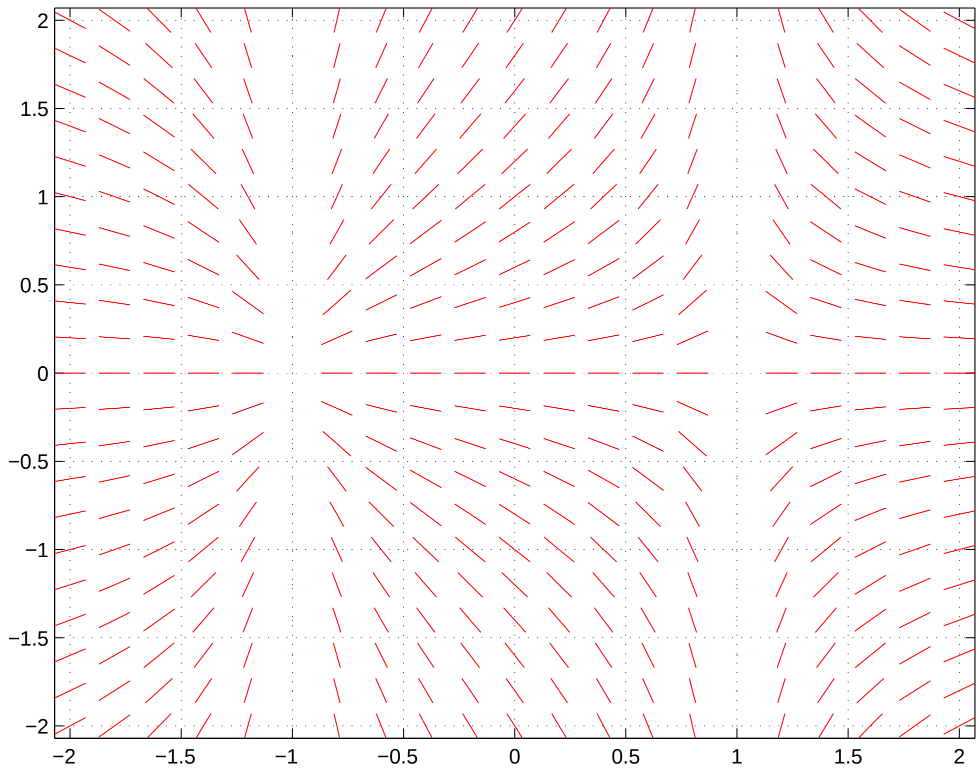
(c) *[5 points].* Sketch the one-parameter family of curves $y(x) = \dfrac{1}{c^2 - x^2}$ for three different positive values of c, (say $c = 0.1$, $c = 1$ and $c = 10$) on the same axes below. Be sure to indicate the behavior of the solution as $x \to \pm c$.

(d) *[5 points].* Use your answers from part (a), part (b) and part (c) to determine the **interval of definition** of the given initial value problem $\dfrac{dy}{dx} = 2xy^2, y(0) = \dfrac{1}{c^2}$ with $c > 0$. Discuss any inconsistencies among your answers.

2. *[25 points total.]* **Direction Fields, Equilibria.**

(a) *[10 points].* Which one of the following differential equations has its direction field represented below? Circle your answer **and** provide an explanation for your selection.

1. $y' = \dfrac{y^2}{y-1}$ **2.** $y' = \dfrac{y}{x^2-1}$ **3.** $y' = \dfrac{y}{y^2-1}$ **4.** $y' = \dfrac{y}{1-x^2}$ **5.** $y' = \dfrac{y}{\sin(\pi x)}$



(b) *[5 points].* Sketch the solution curve that goes through $(0, 1/2)$ on the graph above.

(c) *[10 points].* Indicate and classify the equilibrium solution(s) of the differential equation, or explain why they do not exist.

3. *[20 points total.]* **TRUE or FALSE.**

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) **TRUE or FALSE?** The initial value problem $y' + 2xy = 3x, y(0) = 1$ possesses the one-parameter family of solutions $y(x) = \frac{1}{2}(3 - Ce^{-x^2})$.

(b) **TRUE or FALSE?** The solution of this differential equation $x^2 \dfrac{d^2y}{dx^2} = 2y$ consists of the linear combinations of the linearly independent functions $y_1(x) = x^2$ and $y_2(x) = \dfrac{1}{x}$.

4. *[30 pts. total]* **Undetermined Coefficients, Mathematical Models.**
A mathematical model for the deflection of a uniform beam under a load subject to an axial force is given by the differential equation

$$\frac{d^4y}{dx^4} - k^2\frac{d^2y}{dx^2} = q(x), \qquad 0 < x < L$$

where $y(x)$ is the deflection of the beam, L is the length of the beam, k^2 is proportional to the axial force and $q(x)$ is proportional to the load.

(a) *[10 pts]* Find the homogeneous solution $y_h(x)$ to the differential equation $y^{(4)} - k^2y'' = 0$.

(b) *[10 points]* Let $q(x) = 1$. Find a particular solution y_p to the differential equation $y^{(4)} - k^2y'' = 1$. What is the general solution to this differential equation?

The general solution to the non-homogeneous differential equation $y^{(4)} - k^2 y'' = q(x)$ can be written

$$\begin{aligned} y(x) &= C_1 + C_2 x + C_3 e^{kx} + C_4 e^{-kx} + \frac{1}{k^2}\int q(x)\ x\ dx - \frac{x}{k^2}\int q(x)\ dx \\ &+ \frac{e^{kx}}{2k^3}\int q(x)\ e^{-kx}\ dx - \frac{e^{-kx}}{2k^3}\int q(x)\ e^{kx}\ dx \end{aligned}$$

(c) *[10 points]* Show that the general solution computed in parts (a) and (b) match the general solution given on this page when $q(x) = 1$.

BONUS QUESTION (d) *[10 points]* Confirm that the expression given on this page is indeed a solution of the differential equation $(D^4 - k^2 D^2)y = q(x)$.