FINAL EXAM: Differential Equations

Math 341 ©Prof. R. Buckmire

Assigned: April 29, 2005, noon. Due: May 5, 2005, noon

Name: ______________________

Directions: There are *six* (6) problems on this exam. One problem corresponding to each of the Chapters we discussed in the class (Chapter 1, 2 4,6, 7 and 8) of *Zill*. This is a take-home final exam.

No.	Score	Maximum
1		30
2		30
3		40
4		30
5		40
6		30
Total		**200**

1. *[30 points total.]* **Chapter 1.**

(a) *[10 points].* Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) *[10 points].* Explain why the solution to the initial value problem $y' = 1+y^2, y(0) = 0$ is not defined for every x on the interval $-2 < x < 2$. Why does this not violate the Existence and Uniqueness Theorem?

(c) *[10 points].* Find the largest interval on which the solution to the IVP $y' = 1+y^2, y(0) = 0$ is defined.

2. *[30 points total.]* **Chapter 2.**

(a) *[15 points].* Solve the initial value problem $t\dfrac{dQ}{dt} + Q = t^4 \ln t, \quad y(1) = 1/5.$

(a) *[15 points].* Consider the autonomous differential equation $\dfrac{dy}{dt} = ky^n$ where n is a positive integer and $k \neq 0$. Classify the stationary point at $y = 0$ of the DE for all possible values of k and n.

3. *[40 points total.]* **Chapter 4.**
Our goal is to obtain the equation of a curve $y(x)$ which solves $xy'' + y' + \sqrt{x} = 0$ but is exactly tangential to the x-axis at $x = 1$.
(a) *[10 points].* Use an ansatz of $y = x^m$ to obtain the homogeneous solution
$y = c_1 y_1(x) + c_2 y_2(x)$.

(b) *[10 points].* Use variation of parameters show that the particular solution
$y_p(x) = -\dfrac{2}{3}x^{3/2}$.

(c) *[10 points].* Use the initial conditions given to obtain the equation of the specific curve which satisfies $xy'' + y' + \sqrt{x} = 0$ but is tangential to the x-axis at $x = 1$.

(d) *[10 points].* Sketch a graph of the curve in the space below (or attach a plot obtained from some software program).

4. *[30 pts. total]* **Chapter 6.**
When λ is a known parameter, we have the Laguerre Equation

$$xy'' + (1 - x)y' + \lambda y = 0$$

(a) *5 points.* Show that $x = 0$ is a regular singular point of the Laguerre Equation.

(b) *5 points.* Find and solve the indicial equation of the Laguerre Equation.

(c) *10 points.* Show that the recurrence relation for one of the solutions of this differential equation is $a_n = \dfrac{(n - 1 - \lambda)}{n^2} a_{n-1}$ for $n \geq 1$.

(d) *10 points.* Show that if $\lambda = n$ is a positive integer (say $n = 3$, for example) then all terms past x^n in the power series expansion of the solution $y(x)$ are zero, and thus the Laguerre differential equation has as its solution an n^{th} degree polynomial, known as **a Laguerre Polynomial** $y(x) = a_0 L_n(x)$. Write down $L_0(x)$, $L_1(x)$, $L_2(x)$ and $L_3(x)$.

5. *[40 points total.]* **Chapter 7.**
Our goal in this problem is to obtain the Laplace Transform of a Bessel Function. Consider the initial value problem in **Question 60, Page 314**,

$$ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

(a) *[10 points].* **Confirm** that the exact solution of this initial value problem is Bessel's function of order zero of the first kind $y(t) = J_0(t)$.

(b) *[10 points].* Given that $\mathcal{L}[ty'] = -\frac{d}{ds}\{\mathcal{L}[y']\} = -\frac{d}{ds}\{sY(s) - y(0)\} = -s\frac{dY}{ds} - Y$, show that applying the Laplace Transform to the given initial value problem produces the equation $\mathcal{L}[ty'' + y' + ty] = -(s^2+1)\frac{dY}{ds} - sY(s) = 0$.

(c) *[10 points].* Solve the differential equation in **(b)** for $Y(s)$. You should have an unknown constant $A = Y(0)$ in your answer.

(d) *[10 points].* Since $y(t) = J_0(t)$ is a continuous function of exponential order, it is true that $\lim_{s\to\infty} sY(s) = y(0)$. Use this information to obtain the value of A and show that $\mathcal{L}[J_0(t)] = Y(s) = \dfrac{1}{\sqrt{s^2+1}}$.

6. *[30 points total.]* **Chapter 8**

Consider the homogeneous system, $\frac{d\vec{x}}{dt} = A\vec{x}$, where $A = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix}$.

(a) *[10 points].* Find the eigenvalues of A and their associated eigenvectors.

(b) *[10 points].* Write down the general form of the solution $\vec{x}(t)$.

(c) *[10 points].* Find the solution when $\vec{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$.