Differential Equations

Worksheet 26

TITLE Introducing The Laplace Transform
CURRENT READING Blanchard, 6.1

Homework #11 Assignments due Monday November 16
Section 6.1: 2, 3, 5, 7, 8, 9, 15, 18, 25*.
Section 6.2: 1, 2, 4, 8, 15, 16, 18*.
Homework #12 Assignments due Monday November 23
Section 6.3: 5, 6, 8, 15, 18, 27, 28.
Section 6.4: 1, 2, 6, 7*.

SUMMARY
We introduce a new kind of operator, an integral operator, called the Laplace Transform, which can be used to solve differential equations.

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1. Introducing The Laplace Transform

DEFINITION: **Integral Transform**
If a function $f(t)$ is defined on $[0, \infty)$ then we can define an integral transform to be the improper integral $F(s) = \displaystyle\int_0^\infty K(s,t)f(t)\,dt$. If the improper integral converges then we say that $F(s)$ is the integral transform of $f(t)$. The function $K(s,t)$ is called the **kernel** of the transform. When $K(s,t) = e^{-st}$ the transform is called **the Laplace Transform.**

DEFINITION: **Laplace Transform**
Let $f(t)$ be a function defined on $t \geq 0$. The Laplace Transform of $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

(Note the use of capital letters for the transformed function and the lower-case letter for the input function.) Some people use curly brackets to denote the input, like $\mathcal{L}\{f(t)\}$ but we will use the textbook's notation of square bracket.)

EXAMPLE Let's show that $\mathcal{L}[1] = \dfrac{1}{s}, s > 0$

Exercise
Compute $\mathcal{L}[t]$.

2. Linearity Property of The Laplace Transform

$\mathcal{L}$ is a linear operator, in other words $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

EXAMPLE Let's prove the Laplace Transform possesses the linearity property.

Q: Does every function have a Laplace Transform?
A: Hell, no! (i.e. t^{-1}, e^{t^2} etc do not). Can you think of any others?

DEFINITION: **exponential order**

A function f is said to be of **exponential order** c if there exist constants c, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t \to \infty$ then e^{ct} must approach its limit first, i.e. $\lim_{t\to\infty} \frac{f(t)}{e^{ct}} = 0$.

THEOREM

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $F(s) = \mathcal{L}[f(t)]$ exists for $s > c$ and $\lim_{s\to\infty} F(s) = 0$

This result means that there are functions that clearly can NOT BE Laplace Transforms. These would be functions who do not satisfy the conclusion of the above theorem.

GROUPWORK

Which of the following functions can NOT be Laplace Transforms? Which of the following MIGHT be Laplace Transforms?(HINT: think of the contrapositive of the theorem!)

(a) $\dfrac{s}{s+1}$

(b) $\dfrac{s}{s^2+1}$

(c) $\dfrac{s^2}{s+1}$

(d) $s^2 + 1$

3. Laplace Transforms of Piecewise Continuous Functions

Exercise Find the Laplace Transform of the piecewise function $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$

4. Transforming A Derivative

EXAMPLE We can show that $\mathcal{L}[f'(t)] = sF(s) - f(0)$

THEOREM

If $f, f', f'', f^{(n-1)}, \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and of exponential order c and if $f^{(n)}$ is piecewise continuous on $[0, \infty)$, then $\mathcal{L}[f^{(n)}(t)] = s^n F(s) - \displaystyle\sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$

5. Using Transforms To Solve Differential Equations

EXAMPLE **Blanchard, Devaney & Hall, page 577, #15.** Use the Laplace Transform to solve the initial value problem $\dfrac{dy}{dt} = -y + e^{-2t}, \quad y(0) = 2$

6. The Inverse Laplace Transform

DEFINITION: **Inverse Laplace Transform**

If $F(s)$ represents the Laplace Transform of a function $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$ then the Inverse Laplace Transform of $F(s)$ is $f(t)$, i.e. $\mathcal{L}^{-1}[F(s)] = f(t)$.

Laplace Transforms	
$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\frac{dg}{dt}$	$sG(s) - g(0)$

Inverse Laplace Transforms	
$F(s)$	$f(t) = \mathcal{L}^{-1}[F(s)]$
$\frac{1}{s}$	1
$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
$\frac{1}{s-a}$	e^{at}
$\frac{k}{s^2+k^2}$	$\sin(kt)$
$\frac{s}{s^2+k^2}$	$\cos(kt)$
$\frac{k}{s^2-k^2}$	$\sinh(kt)$
$\frac{s}{s^2-k^2}$	$\cosh(kt)$
$sG(s) - g(0)$	$\frac{dg}{dt}$

Exercise

Compute $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right]$ and $\mathcal{L}^{-1}\left[\frac{1}{s^2+7}\right]$

EXAMPLE

Let's show that $\mathcal{L}^{-1}\left[\frac{-2s+6}{s^2+4}\right] = -2\cos(2t) + 3\sin(2t)$