Math 341 Fall 2014 **SHOW ALL YOUR WORK** Quiz **NINE**

1. The **Gamma Function** $\Gamma(\alpha)$ is defined below as

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,dt, \quad (\alpha > 0).$$

(a) *1 point.* Show that $\Gamma(1) = 1$.

$$\Gamma(1) = \int_0^\infty e^{-t}t^0\,dt = \lim_{b\to\infty}\int_0^b e^{-t}dt = \lim_{b\to\infty} -e^{-t}\Big|_0^b = \lim_{b\to\infty} 1 - e^{-b} = 1$$

(b) *2 points.* Show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t}t^\alpha\,dt = -e^{-t}t^\alpha\Big|_0^\infty - \int_0^\infty -e^{-t}\alpha t^{\alpha-1}dt$$

$u = t^\alpha \quad du = \alpha t^{\alpha-1}$

$dv = e^{-t}dt \quad v = -e^{-t}$

$$= \lim_{b\to\infty} -e^{-b}b^\alpha + e^0 0^\alpha + \alpha\int_0^\infty e^{-t}t^{\alpha-1}dt$$

$$\Gamma(\alpha+1) = 0 + \alpha\Gamma(\alpha)$$

(c) *3 points.* Use the results given in **(a)** and **(b)** to show that $\Gamma(n+1) = n!$, where n is a positive integer. (HINT: use mathematical induction).

Base case: $P(0)$: $\Gamma(0+1) = \Gamma(1) = 1 = 0!$ from part (a)

Inductive step

$P(n) \Rightarrow P(n+1)$

$P(n+1)$: $\Gamma((n+1)+1) = (n+1)\Gamma(n+1) = (n+1)n!$ From $P(n)$

$\Gamma(n+2) = (n+1)!$

(d) *4 points.* Use the previous results (**(a)**, **(b)** and **(c)**) to compute the Laplace transform of t^α (for $\alpha > 0$). In other words, show that $\mathcal{L}[t^\alpha] = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ and $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ when n is a positive integer.

$$\mathcal{L}[t^\alpha] = \int_0^\infty e^{-st}t^\alpha dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha \frac{du}{s} = \int_0^\infty \frac{e^{-u}u^\alpha du}{s^{\alpha+1}}$$

let $u = st \Rightarrow t = \frac{u}{s}$

$du = s\,dt$

$$= \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u}u^\alpha du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = \mathcal{L}[t^\alpha]$$

If $\alpha = n \in \mathbb{Z}$, $\mathcal{L}[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}} \leftarrow$ from (c)

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st}t^n dt = \int_0^\infty e^{-u}\frac{u^n du}{s^{n+1}} = \frac{1}{s^{n+1}}\int_0^\infty e^{-u}u^{(n+1)-1}du = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$