Differential Equations - Fourth Edition
Blanchard - Devaney - Hall

The Analytic, Qualitative, and Numerical Approaches

Our discussion of the three population models in this section illustrates three different approaches to the study of the solutions of differential equations. The **analytic** approach searches for explicit formulas that describe the behavior of the solutions. Here we saw that exponential functions give us explicit solutions to the exponential growth model. Unfortunately, a large number of important equations cannot be handled with the analytic approach; there simply is no way to find an exact formula that describes the situation. We are therefore forced to turn to alternative methods.

One particularly powerful method of describing the behavior of solutions is the **qualitative** approach. This method involves using geometry to give an overview of the behavior of the model, just as we did with the logistic population growth model. We do not use this method to give precise values of the solution at specific times, but we are often able to use this method to determine the long-term behavior of the solutions. Frequently, this is just the kind of information we need.

The third approach to solving differential equations is **numerical.** The computer approximates the solution we seek. Although we did not illustrate any numerical techniques in this section, we will soon see that numerical approximation techniques are a powerful tool for giving us intuition regarding the solutions we desire.

All three of the methods we use have certain advantages, and all have drawbacks. Sometimes certain methods are useful while others are not. One of our main tasks as we study the solutions to differential equations will be to determine which method or combination of methods works in each specific case. In the next three sections, we elaborate on these three techniques.

EXERCISES FOR SECTION 1.1

In Exercises 1 and 2, find the equilibrium solutions of the differential equation specified.

1. $\dfrac{dy}{dt} = \dfrac{y+3}{1-y}$

2. $\dfrac{dy}{dt} = \dfrac{(t^2-1)(y^2-2)}{y^2-4}$

3. Consider the population model

$$\frac{dP}{dt} = 0.4P\left(1 - \frac{P}{230}\right),$$

where $P(t)$ is the population at time t.

(a) For what values of P is the population in equilibrium?

(b) For what values of P is the population increasing?

(c) For what values of P is the population decreasing?

10. The radioactive isotope I-131 is used in the treatment of hyperthyroidism. When administered to a patient, I-131 accumulates in the thyroid gland, where it decays and kills part of that gland.

   (a) Suppose that it takes 72 hours to ship I-131 from the producer to the hospital. What percentage of the original amount shipped actually arrives at the hospital? (See Exercise 7.)
   (b) If the I-131 is stored at the hospital for an additional 48 hours before it is used, how much of the original amount shipped from the producer is left when it is used?
   (c) How long will it take for the I-131 to decay *completely* so that the remnants can be thrown away without special precautions?

11. MacQuarie Island is a small island about half-way between Antarctica and New Zealand. Between 2000 and 2006, the population of rabbits on the island rose from 4,000 to 130,000. Model the growth in the rabbit population $R(t)$ at time t using an exponential growth model

$$\frac{dR}{dt} = kR,$$

where $t = 0$ corresponds to the year 2000. What is an appropriate value for the growth-rate parameter k, and what does this model predict for the population in the year 2010. (For more information on why the population of rabbits exploded, see Review Exercise 22 in Chapter 2.)

12. The velocity v of a freefalling skydiver is well modeled by the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where m is the mass of the skydiver, g is the gravitational constant, and k is the drag coefficient determined by the position of the diver during the dive. (Note that the constants m, g, and k are positive.)

   (a) Perform a qualitative analysis of this model.
   (b) Calculate the terminal velocity of the skydiver. Express your answer in terms of m, g, and k.

Exercises 13–15 consider an elementary model of the learning process: Although human learning is an extremely complicated process, it is possible to build models of certain simple types of memorization. For example, consider a person presented with a list to be studied. The subject is given periodic quizzes to determine exactly how much of the list has been memorized. (The lists are usually things like nonsense syllables, randomly generated three-digit numbers, or entries from tables of integrals.) If we let $L(t)$ be the fraction of the list learned at time t, where $L = 0$ corresponds to knowing nothing and $L = 1$ corresponds to knowing the entire list, then we can form a simple model of this type of learning based on the assumption:

- The rate dL/dt is proportional to the fraction of the list left to be learned.

Since $L = 1$ corresponds to knowing the entire list, the model is

$$\frac{dL}{dt} = k(1 - L),$$

where k is the constant of proportionality.

13. For what value of L, $0 \leq L \leq 1$, does learning occur most rapidly?

14. Suppose two students memorize lists according to the model

$$\frac{dL}{dt} = 2(1 - L).$$

(a) If one of the students knows one-half of the list at time $t = 0$ and the other knows none of the list, which student is learning more rapidly at this instant?

(b) Will the student who starts out knowing none of the list ever catch up to the student who starts out knowing one-half of the list?

15. Consider the following two differential equations that model two students' rates of memorizing a poem. Aly's rate is proportional to the amount to be learned with proportionality constant $k = 2$. Beth's rate is proportional to the square of the amount to be learned with proportionality constant 3. The corresponding differential equations are

$$\frac{dL_A}{dt} = 2(1 - L_A) \quad \text{and} \quad \frac{dL_B}{dt} = 3(1 - L_B)^2,$$

where $L_A(t)$ and $L_B(t)$ are the fractions of the poem learned at time t by Aly and Beth, respectively.

(a) Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having never seen the poem before?

(b) Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having already learned one-half of the poem?

(c) Which student has a faster rate of learning at $t = 0$ if they both start memorizing together having already learned one-third of the poem?

16. The expenditure on education in the U.S. is given in the following table. (Amounts are expressed in millions of 2001 constant dollars.)

Year	Expenditure	Year	Expenditure	Year	Expenditure
1900	5,669	1940	39,559	1980	380,165
1910	10,081	1950	67,048	1990	535,417
1920	12,110	1960	114,700	2000	714,064
1930	30,700	1970	322,935		

(a) Let $s(t) = s_0 e^{kt}$ be an exponential function. Show that the graph of $\ln s(t)$ as a function of t is a line. What is its slope and vertical intercept?

where $c_2 = -3c_1$. Exponentiating we obtain

$$|2000 - 3S| = e^{(-0.03t+c_2)} = c_3 e^{-0.03t},$$

where $c_3 = e^{c_2}$. Note that this means that c_3 is a positive constant. Now we must be careful. Removing the absolute value signs yields

$$2000 - 3S = \pm c_3 e^{-0.03t},$$

where we choose the plus sign if $S(t) < 2000/3$ and the minus sign if $S(t) > 2000/3$. Therefore we may write this equation more simply as

$$2000 - 3S = c_4 e^{-0.03t},$$

where c_4 is an arbitrary constant (positive, negative, or zero). Solving for S yields the general solution

$$S(t) = ce^{-0.03t} + \frac{2000}{3},$$

where $c = -c_4/3$ is an arbitrary constant. We can determine the precise value of c if we know the exact amount of sugar that is initially in the vat. Note that, if $c = 0$, the solution is simply $S(t) = 2000/3$, an equilibrium solution.

EXERCISES FOR SECTION 1.2

1. Bob, Glen, and Paul are once again sitting around enjoying their nice, cold glasses of iced cappucino when one of their students asks them to come up with solutions to the differential equation

$$\frac{dy}{dt} = \frac{y+1}{t+1}.$$

After much discussion, Bob says $y(t) = t$, Glen says $y(t) = 2t + 1$, and Paul says $y(t) = t^2 - 2$.

   (a) Who is right?

   (b) What solution should they have seen right away?

2. Make up a differential equation of the form

$$\frac{dy}{dt} = 2y - t + g(y)$$

that has the function $y(t) = e^{2t}$ as a solution.

3. Make up a differential equation of the form $dy/dt = f(t, y)$ that has $y(t) = e^{t^3}$ as a solution. (Try to come up with one whose right-hand side $f(t, y)$ depends explicitly on both t and y.)

4. In Section 1.1, we guessed solutions to the exponential growth model $dP/dt = kP$, where k is a constant (see page 6). Using the fact that this equation is separable, derive these solutions by separating variables.

In Exercises 5–24, find the general solution of the differential equation specified. (You may not be able to reach the ideal answer of an equation with only the dependent variable on the left and only the independent variable on the right, but get as far as you can.)

5. $\dfrac{dy}{dt} = (ty)^2$

6. $\dfrac{dy}{dt} = t^4 y$

7. $\dfrac{dy}{dt} = 2y + 1$

8. $\dfrac{dy}{dt} = 2 - y$

9. $\dfrac{dy}{dt} = e^{-y}$

10. $\dfrac{dx}{dt} = 1 + x^2$

11. $\dfrac{dy}{dt} = 2ty^2 + 3y^2$

12. $\dfrac{dy}{dt} = \dfrac{t}{y}$

13. $\dfrac{dy}{dt} = \dfrac{t}{t^2 y + y}$

14. $\dfrac{dy}{dt} = t\sqrt[3]{y}$

15. $\dfrac{dy}{dt} = \dfrac{1}{2y + 1}$

16. $\dfrac{dy}{dt} = \dfrac{2y + 1}{t}$

17. $\dfrac{dy}{dt} = y(1 - y)$

18. $\dfrac{dy}{dt} = \dfrac{4t}{1 + 3y^2}$

19. $\dfrac{dv}{dt} = t^2 v - 2 - 2v + t^2$

20. $\dfrac{dy}{dt} = \dfrac{1}{ty + t + y + 1}$

21. $\dfrac{dy}{dt} = \dfrac{e^t y}{1 + y^2}$

22. $\dfrac{dy}{dt} = y^2 - 4$

23. $\dfrac{dw}{dt} = \dfrac{w}{t}$

24. $\dfrac{dy}{dx} = \sec y$

In Exercises 25–38, solve the given initial-value problem.

25. $\dfrac{dx}{dt} = -xt, \quad x(0) = 1/\sqrt{\pi}$

26. $\dfrac{dy}{dt} = ty, \quad y(0) = 3$

27. $\dfrac{dy}{dt} = -y^2, \quad y(0) = 1/2$

28. $\dfrac{dy}{dt} = t^2 y^3, \quad y(0) = -1$

29. $\dfrac{dy}{dt} = -y^2, \quad y(0) = 0$

30. $\dfrac{dy}{dt} = \dfrac{t}{y - t^2 y}, \quad y(0) = 4$

31. $\dfrac{dy}{dt} = 2y + 1, \quad y(0) = 3$

32. $\dfrac{dy}{dt} = ty^2 + 2y^2, \quad y(0) = 1$

33. $\dfrac{dx}{dt} = \dfrac{t^2}{x + t^3 x}, \quad x(0) = -2$

34. $\dfrac{dy}{dt} = \dfrac{1 - y^2}{y}, \quad y(0) = -2$

35. $\dfrac{dy}{dt} = (y^2 + 1)t, \quad y(0) = 1$

36. $\dfrac{dy}{dt} = \dfrac{1}{2y + 3}, \quad y(0) = 1$

37. $\dfrac{dy}{dt} = 2ty^2 + 3t^2 y^2, \quad y(1) = -1$

38. $\dfrac{dy}{dt} = \dfrac{y^2 + 5}{y}, \quad y(0) = -2$

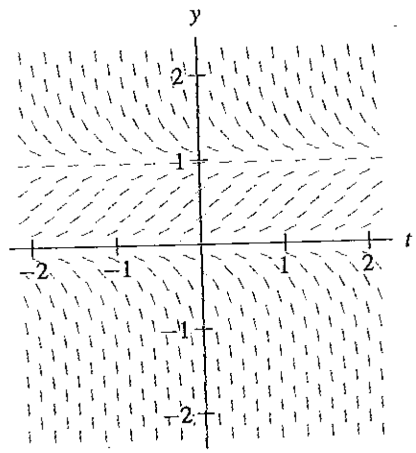
In Exercises 7–10, a differential equation and its associated slope field are given. For each equation,

(a) sketch a number of different solutions on the slope field, and

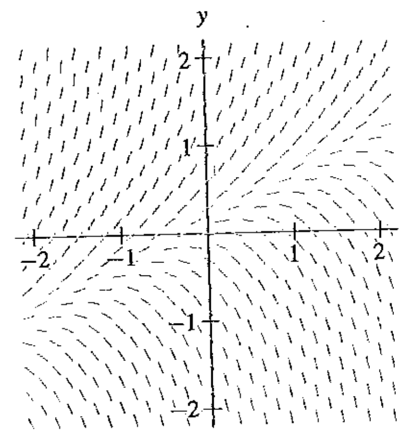
(b) describe briefly the behavior of the solution with $y(0) = 1/2$ as t increases.

You should first answer these exercises without using any technology, and then you should confirm your answer using `HPGSolver`.

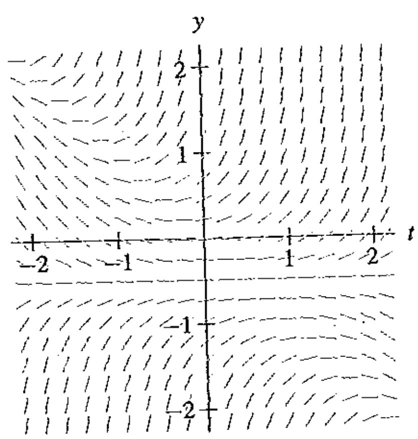
7. $\dfrac{dy}{dt} = 3y(1 - y)$



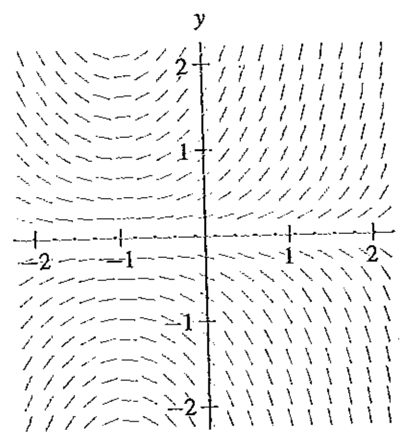
8. $\dfrac{dy}{dt} = 2y - t$



9. $\dfrac{dy}{dt} = \left(y + \frac{1}{2}\right)(y + t)$



10. $\dfrac{dy}{dt} = (t + 1)y$



11. Suppose we know that the function $f(t, y)$ is continuous and that $f(t, 3) = -1$ for all t.

    (a) What does this information tell us about the slope field for the differential equation $dy/dt = f(t, y)$?

    (b) What can we conclude about solutions $y(t)$ of $dy/dt = f(t, y)$? For example, if $y(0) < 3$, can $y(t) \to \infty$ as t increases?

12. Suppose the constant function $y(t) = 2$ for all t is a solution of the differential equation

$$\frac{dy}{dt} = f(t, y).$$

    (a) What does this tell you about the function $f(t, y)$?

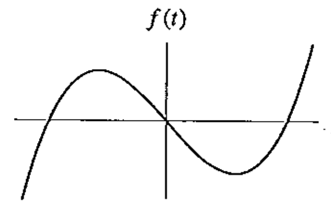
    (b) What does this tell you about the slope field? In other words, how much of the slope field can you sketch using this information?

    (c) What does this tell you about solutions with initial conditions $y(0) \neq 2$?

13. Suppose we know that the graph to the right is the graph of the right-hand side $f(t)$ of the differential equation

$$\frac{dy}{dt} = f(t).$$

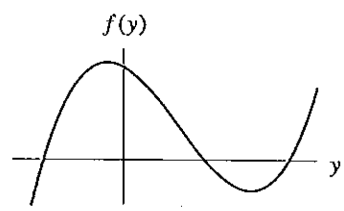
Give a rough sketch of the slope field that corresponds to this differential equation.



14. Suppose we know that the graph to the right is the graph of the right-hand side $f(y)$ of the differential equation

$$\frac{dy}{dt} = f(y).$$

Give a rough sketch of the slope field that corresponds to this differential equation.



15. Consider the autonomous differential equation

$$\frac{dS}{dt} = S^3 - 2S^2 + S.$$

    (a) Make a rough sketch of the slope field without using any technology.

    (b) Using this drawing, sketch the graphs of the solutions $S(t)$ with the initial conditions $S(0) = 1/2$, $S(1) = 1/2$, $S(0) = 1$, $S(0) = 3/2$, and $S(0) = -1/2$.

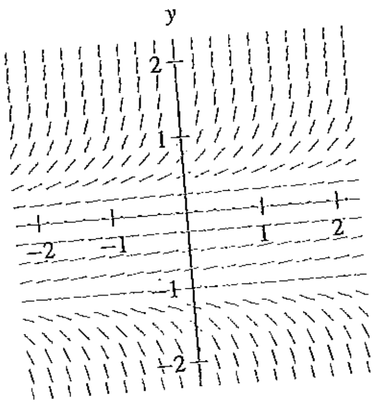
    (c) Confirm your answer using `HPGSolver`.

16. Eight differential equations and four slope fields are given below. Determine the equation that corresponds to each slope field and state briefly how you know your choice is correct. You should do this exercise without using technology.

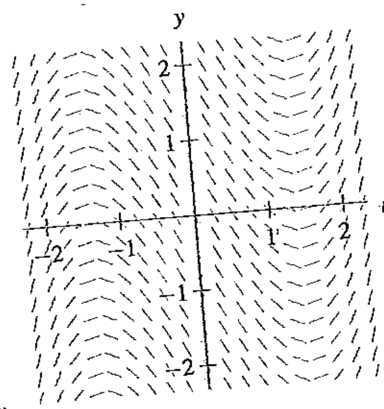
(i) $\dfrac{dy}{dt} = y^2 + y$ (ii) $\dfrac{dy}{dt} = y^2 - y$ (iii) $\dfrac{dy}{dt} = y^3 + y^2$ (iv) $\dfrac{dy}{dt} = 2 - t^2$

(v) $\dfrac{dy}{dt} = ty + ty^2$ (vi) $\dfrac{dy}{dt} = t^2 + t^2 y$ (vii) $\dfrac{dy}{dt} = t + ty$ (viii) $\dfrac{dy}{dt} = t^2 - 2$

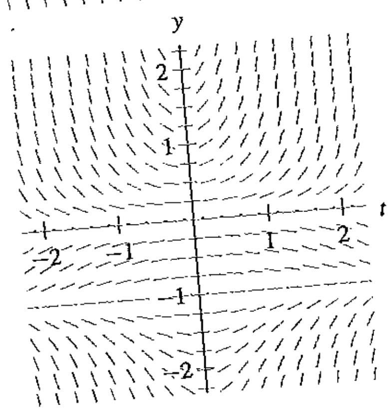
(a)



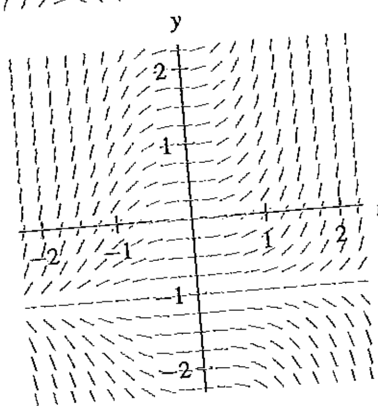
(b)



(c)



(d)



17. Suppose we know that the graph below is the graph of a solution to $dy/dt = f(t)$.

(a) How much of the slope field can you sketch from this information? [*Hint*: Note that the differential equation depends only on t.]

(b) What can you say about the solution with $y(0) = 2$? (For example, can you sketch the graph of this solution?)