FINAL EXAM: Differential Equations

Math 341 Fall 2013

Monday December 9
6:30pm-9:30pm

Name: BUCKMIRE

Directions: Read *all* **ten problems** first before answering any of them. There are **13** pages in this test. This exam is designed to be a 2-hour cumulative exam on the central ideas, techniques and methods of the course. You have three hours to complete the entire exam. **No calculators.** You must show all relevant work to support your answers. You must work alone. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

You may consult a 8.5" by 11" "cheat sheet" with writing on both sides. There is a formula sheet at the end of The Exam which includes all the formulas you should need for Laplace Transforms.

Pledge: I, ________________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
BONUS		10
TOTAL		**100**

1 *[10 points total.]* **Existence and Uniqueness Theorem.**

(a) Suppose $y' = f(x,y)$ where f and $\frac{\partial f}{\partial y}$ are continuous everywhere, i.e. at all (x,y) values. Does this mean that any solution $y(x)$ to the ODE must be continuous everywhere? EXPLAIN YOUR ANSWER.

No.

Counter example: $\left.\begin{array}{l} y' = 1+y^2 \\ y(0) = 0 \end{array}\right\} y(x) = \tan x$

EUT only tells you about existence & uniqueness in a finite region, so you do not know that y(x) will be continuous everywhere, since you don't even know whether y(x) exists everywhere.

~~For~~ Since f and $\frac{\partial f}{\partial y}$ are continuous everywhere, then the solution y(x) must be at least piecewise continuous?

(b) Suppose $y' = xy^{1/3}$. What does the Existence and Uniqueness Theorem allow us to conclude about solutions to the ODE for initial conditions where $y = 0$ for some value of x? EXPLAIN YOUR ANSWER.

$f(x,y) = xy^{1/3}$

$\frac{\partial f}{\partial y} = \frac{1}{3}xy^{-2/3} = \frac{x}{3y^{2/3}}$

When $y=0$ f is continuous so solutions EXIST

When $y=0$ f_y does not exist so uniqueness is NOT guaranteed.

In fact clearly $y=0$ is another sol^n of $y' = xy^{1/3}$

2 *[10 points total.]* **Bifurcation, Phase Lines, Equilibria, Stability.**
Consider the autonomous ordinary differential equation $y' = ay^2 + y^4$ where a is an arbitrary real-valued parameter. Find the equilibria values y^* and compute the bifurcation value a_B. Sketch phase lines for representative values $a < a_B$, $a = a_B$ and $a > a_B$. Clearly label any sinks, sources or nodes. Finally, sketch the bifurcation diagram in the ay^*-plane.

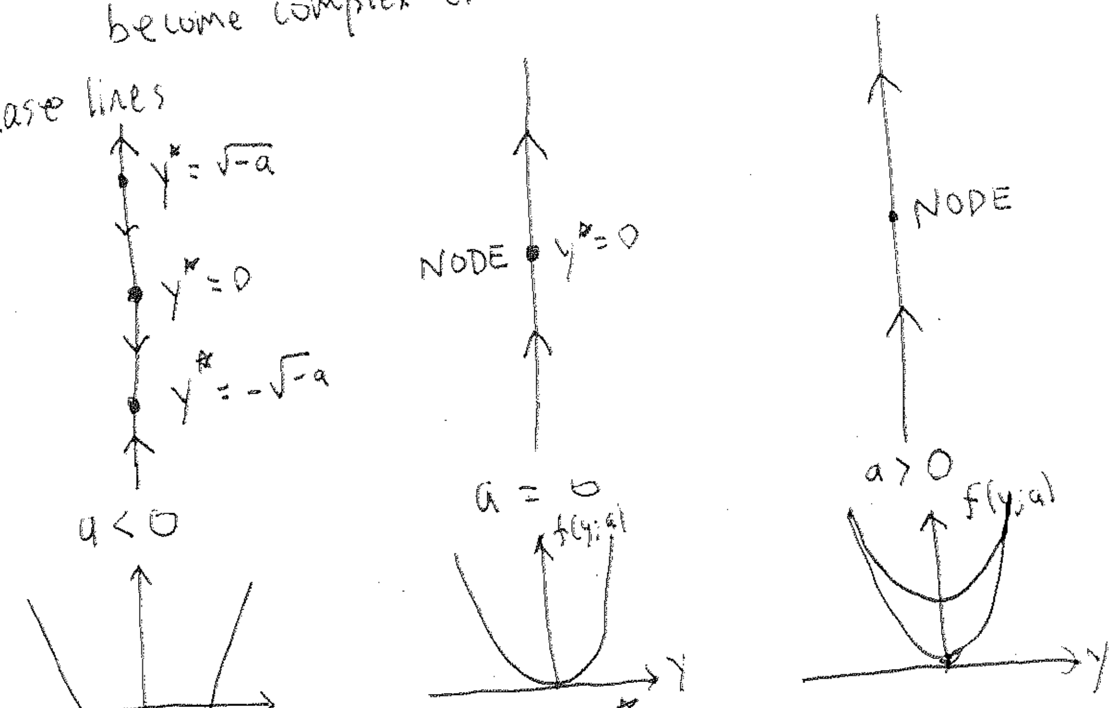
$$y' = f(y;a) = ay^2 + y^4$$

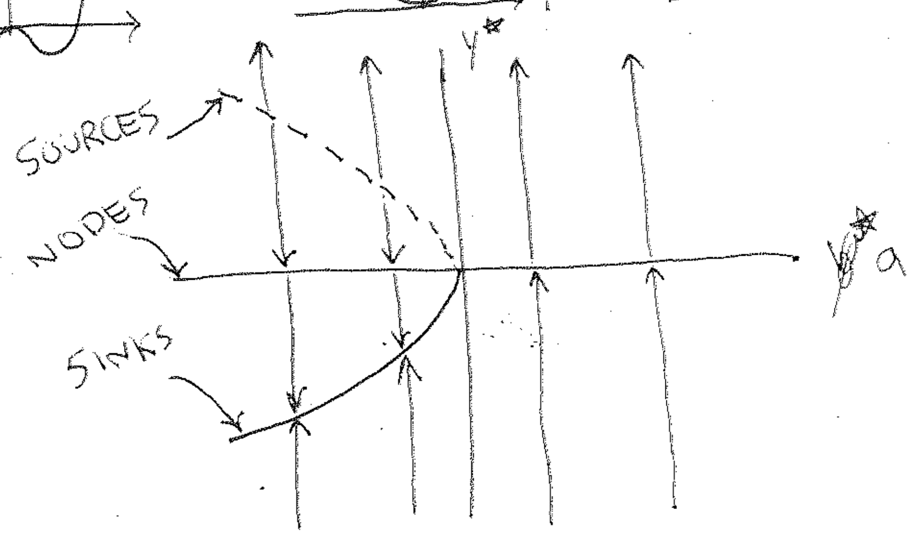
Find equilibria

$$ay^2 + y^4 = 0$$
$$y^2(a + y^2) = 0$$
$$y^* = 0 \quad \text{or} \quad y^2 = -a \Rightarrow y^* = \pm\sqrt{-a}$$

Clearly $a_b = 0$ since when $a > 0$ equilibria will become complex (i.e. will not exist)

Phase lines





3 *[10 points total.]* **Hamiltonian and Gradient Systems.**

Consider $\phi(x,y) = \dfrac{x^2}{2} - \dfrac{x^4}{4} - \dfrac{y^2}{2} + 8$.

(a) Write down a system of differential equations for which the given $\phi(x,y)$ is a Gradient function. CONFIRM your given system is indeed a Gradient system.

$$\dot{x} = \phi_x = x - x^3 = f$$
$$\dot{y} = \phi_y = -y = g$$

Condition on Gradient

$$f_y = g_x$$
$$0 = 0 \checkmark$$

(b) Write down a system of differential equations for which the given $\phi(x,y)$ is a Hamiltonian function. CONFIRM your given system is indeed a Hamiltonian system.

$$\dot{x} = H_y = -y = f$$
$$\dot{y} = -H_x = -x + x^3 = g$$

Condition on Hamiltonian

$$f_x = -g_y$$
$$0 = 0 \checkmark$$

4 *[10 points total.]* **Equilibria of Nonlinear Systems, Linearization, Jacobian, Eigenvalues.**

Consider the following nonlinear system of differential equations

$$\frac{dx}{dt} = -2x + 2x^2$$
$$\frac{dy}{dt} = -3x + y + 3x^2$$

Find and classify all the equlibrium points of the system.

$-2x + 2x^2 = 0$

$2x(-1+x) = 0$

$x = 0$ or $x = 1$

$-3x + 3x^2 + y = 0$

$x = 0, y = 0$

$x = 1, y = 0$

$$J = \begin{pmatrix} -2+4x & 0 \\ -3+6x & 1 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix}$$

$T = -1$

$D = -2$

$D < 0 \Rightarrow$ saddle point near $(0,0)$

$$J(1,0) = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$T = 3$

$D = 2$

$\lambda = \dfrac{3 \pm \sqrt{9 - 4(2)}}{2} = \dfrac{3 \pm 1}{2}$

$= 2, 1$

~~[illegible]~~

source near $(1,0)$

5 *[10 points total.]* **Solving Linear System of ODEs.**

Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \vec{x}$. $\quad \dot{\vec{x}} = A\vec{x}$

$T = -4 = \text{trace}(A)$

$D = \begin{vmatrix} -2 & 3 \\ 3 & -2 \end{vmatrix} = 4 - 9 = -5 \Rightarrow$ origin resembles saddle

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{-4 \pm \sqrt{16+20}}{2} = \frac{-4 \pm \sqrt{36}}{2} = \frac{-4 \pm 6}{2}$$

$\lambda = -5, 1$

$E_{-5} = \text{null}\left\{\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$= \text{null}\{A + 5I\}$

$E_1 = \text{null}\{A - I\}$

$= \text{null}\left\{\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$ where $A\vec{v}_K = \lambda \vec{v}_K$

$= c_1 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

6 *[10 points total.]* **Nonhomogeneous and Homogeneous Problems.**

4 (a) Solve $t\dfrac{dy}{dt} + 2y = 0$. SHOW ALL YOUR WORK!

$$\frac{dy}{dt} = -\frac{2y}{t} \Rightarrow \frac{dy}{y} = \frac{-2dt}{t}$$

$$\ln y = -2\ln t + C$$

$$y = e^{-2\ln t + c}$$

$$= e^{-\ln t^2} \cdot e^{c}$$

$$y = \frac{1}{t^2} \cdot A$$

Separation of Variables

6 (b) Solve $t\dfrac{dy}{dt} + 2y = 2t^2, \quad y(2) = 1$. SHOW ALL YOUR WORK!

$$y' + \frac{2}{t}y = 2t$$

Integrating Factor

$$\mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2 y' + 2ty = 2t^3$$

$$(t^2 y)' = 2t^3$$

$$t^2 y = \int 2t^3$$

$$= 2\frac{t^4}{4} + C$$

$$y = \frac{t^2}{2} + \frac{C}{t^2}$$

$t = 2,\ y = 1$

$$1 = \frac{(2)^2}{2} + \frac{C}{2^2}$$

$$1 = 2 + \frac{C}{4}$$

$$-1 = \frac{C}{4}$$

$$-4 = c$$

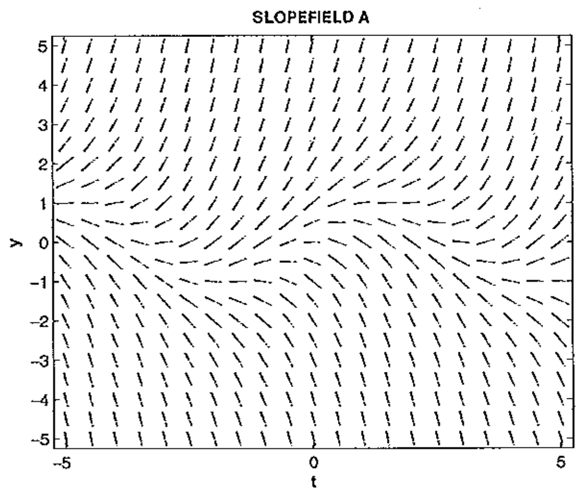
$$y(t) = \frac{t^2}{2} - \frac{4}{t^2}$$

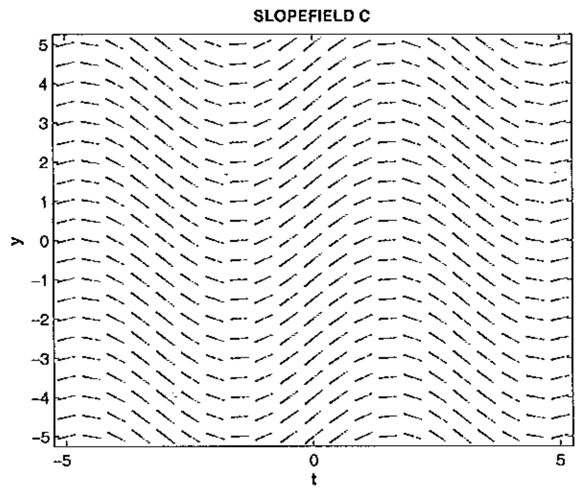
7 *[10 points total.]* **Phase Plane and Slope Fields.**

Consider the following first order differential equations:

1. $\frac{dy}{dt} = y - \sin(t)$ **2.** $\frac{dy}{dt} = \cos(t)$ **3.** $\frac{dy}{dt} = \cos(y)$ **4.** $\frac{dy}{dt} = t - \sin(y)$

Two slope fields are shown below. Match the slope fields with their associated ODE and provide an explanation. You will obtain four times as much credit for the explanation than for the correct slope field selection.





(a). The equation for slope field **A** is 1 $y' = y - \sin(t)$. This is because:

The slope field depends on both t and y since the lines change orientation as one moves vertically (t constant, y changes) and horizontally (y constant, t changes).

When t = 0 the slopes are positive when y > 0 and slopes are negative when y < 0. Thus $y' = y - \sin t$

(b). The equation for slope field **C** is 2 $\frac{dy}{dt} = \cos(t)$. This is because:

Slopefield C only depends on t since the slope does not change as y changes. DE #2 which has this feature. Note that at t = 0, $y' = \cos(0) = 1$, and the slopes look like positive slopes at 45°.

8 *[10 points total.]* **Heaviside Function and Delta Function.**
(a) Solve $y'' + y = \mathcal{H}(t), \quad y(0) = 0, \quad y'(0) = 0.$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\mathcal{H}(t)\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\mathcal{H}(t)\}$$

$$s^2 Y - s y(0) - y'(0) + Y = \frac{e^{-0 \cdot s}}{s}$$

$$(s^2+1) Y = \frac{1}{s}$$

$$Y = \frac{1}{s^2+1} \cdot \frac{1}{s} = \frac{-s}{s^2+1} + \frac{1}{s}$$

$$y(t) = \cancel{\sin(t)} - \cos(t) + 1$$

Check: $y(0) = 1 - \cos(0) = 1 - 1 = 0$ ✓
$y'(0) = \sin(0) = 0$ ✓

(b) Solve $y'' + y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\delta(t)\}$$

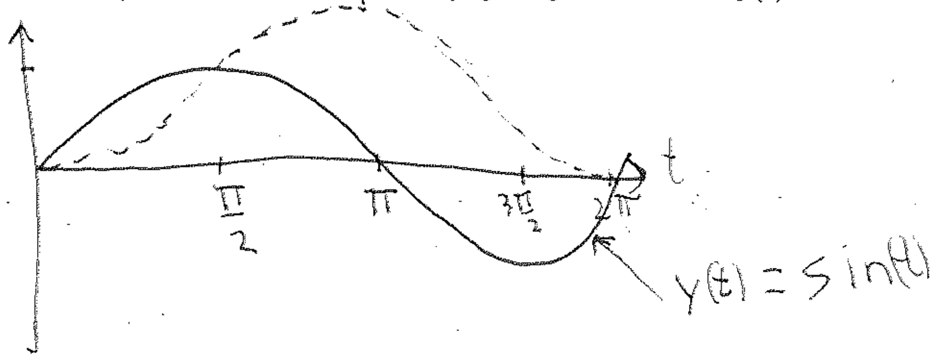
$$(s^2+1) Y = 1$$

$$Y = \frac{1}{s^2+1}$$

$$y(t) = \sin(t)$$

Note the 2^{nd} IC is $y'(0) = 0^-$ which means right before $t=0$, $y' = 0$

(c) Explain the difference between the two different physical situations represented by the two different initial value problems given in (a) and (b) and describe how the solutions to the two problems differ. One way to do this would be by giving a sketch of $y(t)$ for each problem.



9 *[10 points total.]* **Laplace Transforms and Initial Value Problems.**
Consider the following initial value problem, whose exact solution $y(t)$ is the zeroth-order Bessel Function of the First Kind, denoted $J_0(t)$

$$ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

$-\left\{\frac{ds}{ds}\cdot Y + \frac{dY}{ds}\cdot s - \frac{d\{y(0)\}}{ds}\right\}$

Given that $\mathcal{L}[ty'] = -\frac{d}{ds}\{\mathcal{L}[y']\} = -\frac{d}{ds}\{sY(s) - y(0)\} = -s\frac{dY}{ds} - Y$, show that applying the Laplace Transform to the given initial value problem produces the following equation for $Y(s)$.

$$\mathcal{L}[ty'' + y' + ty] = -(s^2+1)\frac{dY}{ds} - sY = 0$$

[HINT: Since you are given $\mathcal{L}[ty']$, how can you use that information to find expressions for $\mathcal{L}[ty'']$ and $\mathcal{L}[ty]$? Think about what multiplication in t-space means when Laplace-transformed into s-space.]

$$\mathcal{L}\{ty\} = -\frac{d}{ds}\mathcal{L}\{y\} = -\frac{dY}{ds}$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2Y - s\right] = -2sY - s^2\frac{dY}{ds} + 1$$

$$\mathcal{L}\{y'\} = sY - 1$$

$$\mathcal{L}\{ty'' + y' + ty\} = -2sY - s^2\frac{dY}{ds} + 1 + sY - 1 - \frac{dY}{ds} = 0$$

$$= (-1 - s^2)\frac{dY}{ds} - sY = 0$$

$$= -(1+s^2)\frac{dY}{ds} = sY$$

$$\frac{dY}{ds} = \frac{-sY}{s^2+1}$$

10 *[10 points total.]* **Separation of Variables.**
Consider the differential equation from **9**,

$$-(s^2+1)\frac{dY}{ds} - sY = 0$$

Let $Y(0) = A$ where A is a (un)known real value. Solve this initial value problem for $Y(s)$. (SEE BONUS PROBLEM to determine A!)

$$\frac{dY}{ds} = \frac{-s}{s^2+1}\,Y$$

$$\frac{dY}{Y} = \frac{-s}{s^2+1}\,ds$$

$$\int \frac{dY}{Y} = -\int \frac{s}{s^2+1}\,ds$$

$$\ln Y = -\frac{1}{2}\ln(s^2+1) + C$$

$$Y = e^{-\frac{1}{2}\ln(s^2+1)+C}$$

$$Y = e^{\ln\left(\frac{1}{\sqrt{s^2+1}}\right)} \cdot e^C$$

$$Y = \frac{A}{\sqrt{s^2+1}}$$

BONUS QUESTION *[10 points total.]*
(a) Given that $y(t)$ and its Laplace Transform $Y(s)$ satisfy the relationship

$$\lim_{s\to\infty} sY(s) = y(0)$$

use this result to obtain the value of the unknown constant A from **10.**

$$\lim_{s\to\infty} s \frac{A}{\sqrt{s^2+1}} = y(0) = 1$$

$$\lim_{s\to} \frac{s}{\sqrt{s^2+1}} A = y(0)$$

$$\lim_{s\to\infty} \frac{1}{\sqrt{1+\frac{1}{s^2}}} A$$

$$1 \cdot A = 1$$

$$A = 1$$

(b) From **9**, **10** and **part (a)** above, what is an explicit functional form for the Laplace Transform of the Zeroth-Order Bessel's Function of the First Kind, i.e. $\mathcal{L}[J_0(t)]$? Recall, $J_0(t)$ satisfies the initial value problem from **9**: $ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Laplace Transforms	
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{t^{m-1}}{(m-1)!}$	$\frac{1}{s^m}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\delta(t-a)$	e^{-as}
$\mathcal{H}(t-a)$ or $u_a(t)$	$\frac{e^{-as}}{s}$

Laplace Transform Formulas	
f'	$sF(s) - f(0)$
f''	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \ldots - f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)\mathcal{H}(t-a)$	$e^{-as}F(s)$
$t^nf(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
$\int_0^t f(\tau)\ d\tau$	$\frac{F(s)}{s}$
$\int_0^t \int_0^r f(\tau)\ d\tau dr$	$\frac{F(s)}{s^2}$
$f * g$	$F(s)G(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\ du$
$f(t+T) = f(t)$	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st}\ f(t)\ dt$