Complex Analysis

Math 312 Spring 1998 MWF 10:30am - 11:25am
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Class 26 **(Monday March 30)**

SUMMARY Applications and Implications of Cauchy's Integral Formula
CURRENT READING Brown & Curchill pages 123-125
NEXT READING Brown & Curchill pages 125-129

Applications of Cauchy's Integral Formula

Let C be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing C and z_0 is **any point inside** of C, then

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

and

$$\int_C \frac{f(z)}{(z-z_0)^m} dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0)$$

These two results lead to a number of other results. Actually, the two formulas are just restatement of one formula, known as the *generalized Cauchy Integral Formula*. Can you see how the first expression (**CIF**) is just a special case ($m =$??) of the second one?

Examples

We have rewritten the integral formulas in the way above so that we can use them to actually evaluate integrals. Let's to do the following two.

$\oint_C \frac{e^{5z}}{z^3}\, dz =$

(where C is $|z| = 1$ traversed once clockwise)

$\int_C \frac{2z+1}{z(z-1)^2}\, dz =$

(where C is given in the sketch)

There are numerous theorems which directly follow from Cauchy's Integral Formula. I have listed a *few* of the more famous ones below...

Implications of Cauchy's Integral Formula

Morera's Theorem

If $f(z)$ is continuous in a simply-connected region R and if $\oint_C f(z)dz = 0$ around *every* simple closed curve C in R, then $f(z)$ is analytic in R.
(NOTE: Morera's Theorem is the converse of the Cauchy-Goursat theorem.)

Cauchy's Inequality

If $f(z)$ is analytic inside and on a circle of radius r and centered at $z = z_0$ then

$$|f^{(n)}(z_0)| \leq \frac{M \cdot n!}{r^n} \qquad\qquad n = 0, 1, 2, \ldots$$

where M is an upper bound on $|f(z)|$ on C

Liouville's Theorem

Suppose that for all z in the entire complex plane, if $f(z)$ is analytic and bounded, (i.e. $|f(z)| < M$ for some real constant M) then $f(z)$ must be a constant.

Fundamental Theorem of Algebra

Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has atleast one root.

Gauss' mean value theorem

If $f(z)$ is analytic inside and on a circle C with center z_0 and radius r then $f(z_0)$ is the mean of the values of $f(z)$ on C, namely

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

Maximum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on C.

Minimum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve C and $f(z) \neq 0$ inside C, then the minimum value of $|f(z)|$ occurs on C.

The Argument Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Then

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = N - P$$

where N and P are the number of zeroes and poles of $f(z)$ inside C

Rouche' Theorem

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside of C.