Complex Analysis

Math 312 Spring 1998 MWF 10:30am - 11:25am
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Class 20 **(Wednesday March 4)**

SUMMARY Introduction to Contour Integration
CURRENT READING Brown & Curchill, pages 95-104
NEXT READING Brown & Curchill pages 104-122

Exercise
First, let's recall how to integrate complex functions of a **real** variable. Compute the following:

$$(a)\int_1^2 \frac{-i}{t^2} + (t+2i)^3\, dt \qquad\qquad (b)\int_0^\infty e^{-z^2 t}\, dt$$

Contour Integration
Integration of a complex function of a **complex** variable is performed on a set of connected points from, say, z_1 to z_2. It is a **contour integral**. Given a contour C defined as $z(t)$ for $a \le t \le b$ where $z_1 = z(a)$ and $z_2 = z(b)$, an integral of a complex function of a complex variable $f(z)$ is written as

$$\int_C f(z)dz \qquad \text{or} \qquad \int_{z_1}^{z_2} f(z)\, dz$$

Let $f(z)$ be piecewise continuous on $z(t)$. If C is a **contour** then $z'(t)$ is piecewise continuous on $a \le t \le b$ and we can redefine the integral of $f(z)$ along C as:

$$\int_C f(z)\, dz = \int_a^b f[z(t)]z'(t)\, dt$$

Examples
Compute $\int_C \text{Im } z\, dz$ where C is a directed line segment from $z = 0$ to $z = 1 + 2i$
ALGORITHM: (steps to be taken to complete the process of contour integration)
1: write down a parametricization for C, $z(t)$
2: Convert the integral into an integral in (real) t variables
3: Integrate!

GROUPWORK

Compute $\int_C 2\overline{z}^2 \, dz$ where C is a directed line segment from $z = 2$ to $z = -2$. (Sketch the contour and evaluate the integral.)

Also evaluate $\int_C 2\overline{z}^2 \, dz$, this time using C being a counterclockwise circular arc from $z = 2$ to $z = -2$. (Sketch the contour and then evaluate the integral.)

Also evaluate $\int_C 2\overline{z}^2 \, dz$, this time using C being a clockwise circular arc from $z = 2$ to $z = -2$. (Sketch the contour and then evaluate the integral.)

Question

Does the value of a contour integral depand on the path taken?

Exercise

Show that

$$\int_{C_r} (z - z_0)^n \, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

where n is any integer and C_r is a circle of radius r around z_0 (what is the equation of such a shape?) traversed **once** in the counter-clockwise direction. How will our results change if we reverse the direction of travel along the contour (i.e. move in a clockwise direction)?