Complex Analysis

Class 17 **(Friday February 20)**

SUMMARY Linear transformations, inversion mappings, and bilinear transformations
CURRENT READING Brown & Curchill, pages
NEXT READING Brown & Curchill pages

Linear Transformation

Let us re-consider the idea that functions of a complex variable $w = f(z)$ represent a mapping from the complex z=(x,y) plane to the complex $w = (u, v)$ plane

Rotation: Rotation by an angle α
$w = f_1(z) = e^{i\alpha}z$

Scaling: Magnification / reduction by a factor of $|a|$
$w = f_2(z) = az$ (a is a real number)

Translation: Shifting by a vector of size (Re(B), Im(B))
$w = f_3(z) = z + B$

What do all all these transformations have in common? What properties of the image get preserved?

Examples

Consider the circle $C_1 : |z - 1| = 1$. Find a series of transformations which map C_1 onto $C_2 : |z - \frac{3}{2}i| = 2$

Sketch the series of transformations in the space below.

Using the relationship between *composition of functions* and *successive transformations* write down a single function which transforms C_1 into C_2

So, any **linear transformation** can be written as a composition of __________________________ , __________________________ and __________________________

The Inversion Mapping $\frac{1}{z}$

The function $w = \frac{1}{z}$ establishes a one-to-one correspondence between the nonzero points of the z **and** w planes.
Remember $|z|^2 = z\overline{z}$, so $w = 1/z$ can be treated as two successive mappings

$$Z = \frac{1}{|z|^2} z, \qquad w = \overline{Z}$$

These mappings represent a ________________________ followed by a ________________________.

GROUPWORK

Think about the points

a. $(|z| > 1)$: exterior to the circle $|z| = 1$

b. $(|z| < 1)$: interior to the circle $|z| = 1$

c. ON the circle $|z| = 1$

Where do each of these sets get mapped to in the w-plane using the "inversion transformation?" (To answer this question you should pick a point in each one of these sets and see where it is mapped under the $w = 1/z$ transformation. You may want to split this job among the members of the group.)

What is the image of the point $z = 0$ under the inversion mapping? What is the image of the point at infinity under the inversion mapping?
To answer these questions you should recall how we deal with complex limits involving ∞

$$\lim_{z\to\infty} f(z) = w_0 \Longleftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

PREGNANT PAUSE:

Take 2 minutes and think about the concepts on this page.
READ through the worksheet and formulate any questions you may have.

My Question is:

Properties of the Inversion Mapping

The mapping $w = 1/z$ maps the extended complex plane to itself on a one-to-one basis.
The mapping $1/z$ transforms *circles and lines* into *circles and lines*

Lines passing thru the origin	$\longmapsto$	Lines passing thru the origin
Lines NOT passing thru the origin	$\longmapsto$	Circles passing thru the origin
Circles passing thru the origin	$\longmapsto$	Lines NOT passing thru the origin
Circles NOT passing thru the origin	$\longmapsto$	Circles NOT passing thru the origin

Bilinear Transformations
Consider transformations of the form

$$w = f(z) = \frac{az+b}{cz+d}$$

They are also known as **bilinear transformations** or **M$\ddot{o}$bius transformations**. It;s easy to see that you can re-write this to produce an expression of the form

$$Azw + Bz + Cw + D = 0$$

where A, B, C and D can be expressed in terms of a, b, c and d.

Bilinear Transformation as composite mapping
Notice that if $c = 0$ then our bi-linear transformation (linear in z and w) becomes just a linear transformation in z.
If $c \neq 0$ then we can re-write $w = f(z)$ as

$$w = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{cz+d}$$

To show this, all we have to do is remember polynomial division:

If we look at the linear fractional transformation this way, we can see that it can be written as a composition of two linear transformations and an inverse mapping.

$$w = cz+d, \qquad w_1 = \frac{1}{w}, \qquad w_2 = \frac{a}{c} + \frac{bc-ad}{c}w_1$$

Find the composition of these three mappings above, so that $w_2 = T(z)$, and by so doing, show that T is a "LFT."

Thus LFTs can be thought of as a ____________________ followed by a ____________________ followed by a ____________________.

Therefore, we know that LFT's map *circles* and *lines* to ____________________ and ____________________

Properties of Linear Fractional Transformations

Let f be a Mӧbius transformation. Then

- f can be expressed as the composition of a finite number of rotations, translations, magnifications and inversions
- f maps the *extended complex-plane* to itself
- f maps the class of circles and lines to circles and lines
- f is **conformal** (i.e. $f'(z) \neq 0$) at every point besides its pole

Poles and Fixed Points
A **pole** (regular singularity) of a function is a point z_0 where $\lim_{z \to z_0} f(z) = \infty$
A **fixed point** of a function $f(z)$ is a point iz_0 such that $f(z_0) = z_0$. That is the point gets mapped to the same spot in the w-plane.

Find the poles of $T(z) = \dfrac{az+b}{cz+d}$. How many poles does it have? How many fixed points does it have? (These answers should depend on a, b, c and d.)

If a line or circle passes thru the pole of T then it must be mapped to a shape that goes thru the point at infinty. **Why?** What kind of shape would do that?

So, if a line or circle does NOT pass thru the pole of T it must get mapped to what kind of shape?

Where does T map the point at infinity?

Inverses of LFTs
Since T is a one-to-one mapping on the extended complex plane, it has an inverse. If you solve $w = T(z)$ so that $z = T^{-1}(w)$, then

$$T^{-1}(w) = \frac{-dw+b}{cw-a}, \qquad (ad - bc \neq 0)$$

Note that T^{-1} is also an LFT. In general, if S and T are two LFTs, then $S(T(z))$ is also an LFT.

Example

Find the image of the *interior* of the circle $C : |z-2| = 2$ under the LFT given by $w = f(z) = \dfrac{z}{2z-8}$ Sketch the image and pre-image of C under $w = f(z)$