Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class 15 **(Friday February 20)**

SUMMARY Linear transformations, inversion mappings, and bilinear transformations
CURRENT READING Brown & Curchill, pages
NEXT READING Brown & Curchill pages

Linear Transformation

Let us re-consider the idea that functions of a complex variable $w = f(z)$ represent a mapping from the complex z=(x,y) plane to the complex $w = (u, v)$ plane

Rotation: Rotation by an angle α
$w = f_1(z) = e^{i\alpha}z$

Scaling: Magnification / reduction by a factor of $|a|$
$w = f_2(z) = az$ (a is a real number)

Translation: Shifting by a vector of size (Re(B), Im(B))
$w = f_3(z) = z + B$

What do all all these transformations have in common? What properties of the mage get preserved?

Examples

Consider the circle $C_1 : |z - 1| = 1$. Find a series of transformations which map C_1 onto $C_2 : |z - \frac{3}{2}i| = 2$
Sketch the series of transformations on the space below.

Using the relationship between *composition of functions* and *successive transformations* write down a single function which transforms C_1 into C_2

So, any **linear transformation** can be written as a composition of____________________ ,
____________________ and ____________________

The Inversion Mapping $\frac{1}{z}$

The function $w = \frac{1}{z}$ establishes a one-to-one correspondence between the nonzero points of the z **and** w planes.
Remember $|z|^2 = z\overline{z}$, so $w = 1/z$ can be treated as two successive mappings

$$Z = \frac{1}{|z|^2} z, \qquad w = \overline{Z}$$

These mappings represent a ____________________ followed by a ____________________.
Think about the points

a. $(|z| > 1)$: exterior to the circle $|z| = 1$

b. $(|z| < 1)$: interior to the circle $|z| = 1$

c. ON the circle $|z| = 1$

Where do each of these sets get mapped to in the w-plane using the "inversion transformation?" [To answer this question you should pick a point in each one of these sets and see where it is mapped under the $w = 1/z$ transformation]

What about the point $z = 0$? What does "infinity" mean in the realm of complex numbers?

Now that we know about the point at infinity we can take the following limits

$$\lim_{z\to\infty} f(z) = w_0 \Longleftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

PREGNANT PAUSE: Take 2 minutes and think about the concepts on this page.

Properties of the Inversion Mapping

The mapping $w = 1/z$ maps the extended complex plane to itself on a one-to-one basis.
The mapping $1/z$ transforms *circles and lines* into *circles and lines*

Lines passing thru the origin	$\longmapsto$	Lines passing thru the origin
Lines NOT passing thru the origin	$\longmapsto$	Circles passing thru the origin
Circles passing thru the origin	$\longmapsto$	Lines NOT passing thru the origin
Circles NOT passing thru the origin	$\longmapsto$	Circles NOT passing thru the origin

Bilinear Transformations

Consider transformations of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

They are also known as **bilinear transformations** or **M$\ddot{o}$bius transformations**. Show that you can re-write this to produce an expression of the form

$$Azw + Bz + Cw + D = 0$$

Find values for A, B, C and D in terms of a, b, c and d.

Notice that if $c = 0$ then our bi-linear transformation (linear in z and w) becomes just a linear transformation in z.

If $c \neq 0$ then we can re-write $w = f(z)$ as

$$w = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d}$$

If we look at the linear fractional transformation this way, we can see that it can be written as a composition of two linear transformations and an inverse mapping.

$$w = cz + d, \qquad w_1 = \frac{1}{w}, \qquad w_2 = \frac{a}{c} + \frac{bc - ad}{c} w_1$$

Find the composition of these three mappings above, so that $w_2 = T(z)$, and by so doing, show that T is a "LFT"

Therefore, we know that LFT's map *circles* and *lines* to ________________________ and ________________________

Properties of Linear Fractional Transformations

Let f be a M$\ddot{o}$bius transformation. Then

- f can be expressed as the composition of a finite number of rotations, translations, magnifications and inversions
- f maps the *extended complex-plane* to itself
- f maps the class of circles and lines to circles and lines
- f is **conformal** (i.e. $f'(z) \neq 0$) at every point besides its pole

A **pole** (regular singularity) of a function is a point z_0 where $\lim_{z\to z_0} f(z) = \infty$
Find the poles of

$$T(z) = \frac{az+b}{cz+d}$$

If a line or circle passes thru the pole of T then it must be mapped to a shape that goes thru the point at infinty. **Why?** What kind of shape would do that?

So, if a line or circle does NOT pass thru the pole of T it must get mapped to what kind of shape?

Where does T map the point at infinity to?

Since T is a one-to-one mapping on the extended complex plane, it has an inverse. If you solve $w = T(z)$ so that $z = T^{-1}(w)$, then

$$T^{-1}(w) = \frac{-dw+b}{cw-a}, \qquad (ad - bc \neq 0)$$

Note that T^{-1} is also an LFT. In general, if S and T are two LFTs, then$S[T(z)]$ is also an LFT.

Example

Find the image of the *interior* of the circle $|z-2| = 2$ under the LFT

$$w = f(z) = \frac{z}{2z-8}$$

Sketch the image and pre-image of C under $w = f(z)$