# Complex Analysis

Math 312 Spring 1998 | MWF 10:30am - 11:25am
**Buckmire** | Fowler 112

*Class 14* **(Friday February 13)**

**SUMMARY** Complex Exponents $z^c$ and $c^z$ and More on Branch Cuts
**CURRENT READING** Brown & Curchill, pages 75-81
**NEXT READING** Brown & Curchill pages 81-85

**Branching: Producing Single-valued functions from Multiple-valued ones**
A single-valued function $F(z)$ is said to be a *branch* of a multiple-valued function $f(z)$ in a domain $D$ if $F(z)$ is single-valued and analytic in $D$ and has the property that for each $z \in D$, the value $F(z)$ is one of the values of $f(z)$

Branch cuts do not have to be along the $x$-axis, they can be any line which when removed from the domain of definition of the multi-valued function, produce a single valued function.

**Example**
Determine the domain of analyticity for the function $f(z) = \text{ Log }(3z - i)$ and compute $f'(z)$ What is $f(i)$? What about $f'(i)$?

(HINT: think of it as $\text{ Log }(w)$. Where in the $w$-plane is $\text{ Log }(w)$ not analytic?)

**Other branches of** $\log z$
One can define other analytic branches of $\log z$ by choosing different branch cuts.
The usual way to do this is to make the branch cut along $\theta = \alpha$ starting at the origin, so that

$$\log z = \ln|z| + i\theta, \qquad \text{where } \alpha < \theta \le \alpha + 2\pi$$

These branches of $\log z$ can be denoted $\mathcal{L}_\alpha$, where $\theta = \alpha$ is the branch cut.

A **branch point** of a function $f$ is a point which is common to all branch cuts of $f$. So, 0 is a branch point of $\log z$

GROUPWORK

Now consider $\text{Log}\,(z^2+1)$. This is a much more complicated function.
How does it differ from $\log(z^2+1)$ or $\mathcal{L}_\alpha(z^2+1)$?
What are the equations that its branch cut will satisfy?
Sketch the branch cut for $\text{Log}\,(z^2+1)$ on the axes below.

## Roots of Complex Numbers (Reprise)

From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in Z$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right)$$

But

$$\begin{aligned}
\exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln|z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in Z)\\
&= |z|^{1/n}\exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right)\\
&= |z|^{1/n}\exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z
\end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where $k$ is restricted to $0, 1, 2, \ldots, n-1$ . Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when $k$ can be any integer and $n$ is fixed?] What do you think happens i f we try and raise a complex number to something besides an integer or rational number? How will we deal with **complex exponents**?

**Complex Exponents**

If $z \neq 0$ and $c \in \mathcal{C}$, the function $z^c$ is defined as

$$z^c = \exp(\log z^c) = \exp(c \log z)$$

Since $\log z$ is a multi-valued function, $z^c$ will have multiple values. How many values depends on the nature of $c$.

$$z^c = \begin{cases} z^{n/m} & \text{if } c \text{ is rational, i.e. } n/m \quad \text{finite number of values (m)} \\ z^n & \text{if } c = n\text{, where } n \text{ is an integer} \quad \text{single value} \\ z^c & \text{all other complex numbers} \quad \text{infinite number of values} \end{cases}$$

**Example**
Show that $i^i$ is purely real.

GROUPWORK
Compute the following:
(a) $(0.5 - \frac{\sqrt{3}}{2}i)^3 =$

(b) $(-1)^{2/3} =$

(c) $(1+i)^{1-i} =$

**Derivatives of $z^c$ and $c^z$**

If you choose a branch of $z^c$ which is analytic on an open set, then

$$\frac{d}{dz}(z^c) = cz^{c-1}$$

where the branch of the log used in evaluating $z^c$ is the same branch used in evaluating $z^{c-1}$

Similarly, we can define the *complex exponential function with base* $c$

$$c^z = \exp(z \log c)$$

If a single value of $c$ is chosen, then $c^z$ is an entire function such that

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\exp(z \log c) = c^z \log c$$

**Examples**

If $f(z) = (1+i)^z$, Find $f'(1-i)$ (Use $\mathcal{L}_0$)

If $g(z) = z^{(1-i)}$, Find $f'(1+i)$ (Use $\mathcal{L}_0$)

**Branch cuts**

A **branch cut** is a set of points on which we must select a particular value of a multi-valued function in order to construct a single-valued function. For example, the set of points $\text{Im } z = 0 \cap \text{ Re } z \leq -$ is the branch cut on which we select the argument function to have the singular value of $\pi$.

Let us look at $\text{Log }(3z - i)$ again. First, let's consider $\text{Log }(w)$.

Where in the $w$-plane is $\text{Log }(w)$ not analytic?

What are the set of points where $\text{Re}(w) < 0$ and $\text{Im } w = 0$ if $w = 3z - i$?

Draw these points on the axes below.

**Branch chasing**

Other functions have branch cuts, too, of course. We usually have to *choose* the branch cut so that our function in question is analytic at a required point.

<u>GROUPWORK</u>

1: Define a branch of $f(z) = (z^2 - 1)^{1/2}$ so that it is analytic on the exterior of the unit circle $|z| > 1$

(a) First write down the *principal branch* of $f(z)$. Where are its branch cuts? [What equations/inequalities do you have to solve?]

(b) Sketch the branch cuts on the axes below.

(c) What if we redefine $(z^2 - 1)^{1/2} = z(1 - 1/z^2)^{1/2}$? Where are the branch cuts of the principal branch of our new function? Sketch them below

(d) Write down the function which solves our problem #1

The problem of "branch chasing" for complicated functions can be extremely tedious but it is necessary if we want them to be analytic. Thankfully, it is seldom necessary for our elementary applications.