Complex Analysis

Math 312 Spring 1998 MWF 10:30am - 11:25am
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Class 13 **(Wednesday February 11)**

SUMMARY Complex Logarithms and the debut of branch cuts
CURRENT READING Brown & Curchill, pages 75-81
NEXT READING Brown & Curchill pages 81-85

The Complex Logarithm $\log z$
Let us define $w = \log z$ as the inverse of $z = e^w$
But we know that $\exp[\ln|z| + i(\theta + 2n\pi)] = z$, where $n \in Z$, from our knowledge of the exponential function.
So we can define

$$\log z = \ln|z| + i \arg z = \ln|z| + i \text{ Arg } z + 2n\pi i = \ln r + i\theta$$

where $r = |z|$ as usual, and θ is the argument of z
If we only use the principal value of the argument, then we define the principal value of $\log z$ as $\text{Log } z$, where

$$\text{Log } z = \ln|z| + i \text{ Arg } z = \text{ Log } |z| + i \text{ Arg } z$$

Examples
Compute $\text{Log } (-2) and \text{ Log } (2)$, $\log(-2)$, and $\log(2)$, $\text{Log } (-4)$ and $\log(-4)$

Logarithmic Identities
$z = e^{\log z}$ but $\log e^z = z + 2k\pi i$ (Is this a surprise?)
$\log(z_1 z_2) = \log z_1 + \log z_2$
$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
However these do not neccessarily apply to the principal branch of the logarithm, written as $\text{Log } z$.

Log z: the Principal Branch of $\log z$

Log z is a single-valued function and is analytic in the domain D^* consisting of all points of the complex plane *except for those lying on the nonpositive real axis,* where

$$\frac{d}{dz}\text{ Log } z = \frac{1}{z}$$

Sketch the set D^* and convince yourself that it is an open connected set.
(Ask yourself: Is every point in the set an interior point?)

The set of points $\text{Re } z \leq 0 \cap \text{ Im } z = 0$ is a line of discontinuities known as a **branch cut**. By putting in a branch cut we say that we "construct Log z from $\log z$." Why can we not evaluate $\log z$ along the entire positive x-axis?

Analyticity of Log z

We can use a version of the Cauchy-Riemann Equations in polar coordinates to help us investigate the analyticity of Log z

Why don't we investigate the analyticity of $\log z$?

If $x = r\cos\theta$ and $y = r\sin\theta$ one can trewrite $f(z) = u(x,y) + iv(x,y)$ into $f = u(r,\theta) + iv(r,\theta)$ in that case, the CREs become:

$$u_r = \frac{1}{r}v_\theta, \qquad v_r = -\frac{1}{r}u_\theta$$

and the expression for the derivative $f'(z) = u_x + iv_x$ can be re-written

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

GROUPWORK

Using this information, show that Log z is analytic and that $\frac{d}{dz}$ Log $z = \frac{1}{z}$.
(HINT: You will need to write down $u(r,\theta)$ and $v(r,\theta)$ for Log z)

Branching: Producing Single-valued functions from Multiple-valued ones
A single-valued function $F(z)$ is said to be a *branch* of a multiple-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and analytic in D and has the property that for each $z \in D$, the value $F(z)$ is one of the values of $f(z)$

Branch cuts do not have to be along the x-axis, they can be any line which when removed from the domain of definition of the multi-valued function, produce a single valued function.

Example
Determine the domain of analyticity for the function $f(z) = \text{ Log }(3z - i)$ and compute $f'(z)$ What is $f(i)$? What about $f'(i)$?

Other branches of $\log z$
One can define other analytic branches of $\log z$ by choosing different branch cuts.
The usual way to do this is to make the branch cut along $\theta = \alpha$ starting at the origin, so that

$$\log z = \ln |z| + i\theta, \qquad \text{where } \alpha < \theta < \alpha + 2\pi$$

These branches of $\log z$ can be denoted $\mathcal{L}_\alpha$, where $\theta = \alpha$ is the branch cut.

A **branch point** of a function f is a point which is common to all branch cuts of f. So, 0 is a branch point of $\log z$
Roots of Complex Numbers (Reprise)
From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in Z$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n} \log z\right)$$

But

$$\begin{aligned}
\exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln|z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in Z)\\
&= |z|^{1/n} \exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right)\\
&= |z|^{1/n} \exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z
\end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where k is restricted to $0, 1, 2, \ldots, n-1$. Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when k can be any integer and n is fixed?] What do you think happens if we try and raise a complex number to something besides an integer or rational number? How will we deal with **complex exponents**?

Complex Exponents

If $z \neq 0$ and $c \in \mathcal{C}$, the function z^c is defined as

$$z^c = \exp(c \log z)$$

Since $\log z$ is a multi-valued function, z^c will have multiple values. How many values depends on the nature of c.

$$z^c = \begin{cases} z^{n/m} & \text{if } c \text{ is rational, i.e. } n/m \quad \text{finite number of values (m)} \\ z^n & \text{if } c = n, \text{ where } n \text{ is an integer} \quad \text{single value} \\ z^c & \text{all other complex numbers} \quad \text{infinite number of values} \end{cases}$$

Examples

Compute the following:

(a) $(1+i)^{1-i} =$

(b) $(0.5 - i)^3 =$

(c) $(-1)^{2/3} =$

Differentiating

If you choose a branch of z^c which is analytic on an open set, then

$$\frac{d}{dz}(z^c) = cz^{c-1}$$

where the branch of the log used in evaluating z^c is the same branch used in evaluating z^{c-1}

Similarly, we can define the *complex exponential function with base* c

$$c^z = \exp(z \log c)$$

If a single value of c is chosen, then c^z is an entire function such that

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\exp(z \log c) = c^z \log c$$

Examples

If $f(z) = (1+i)^z$, Find $f'(1-i)$ (Use the principal branch)

If $g(z) = z^(1-i)$, Find $f'(1+i)$ (Use the principal branch)