Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class 12 **(Monday February 9)**

SUMMARY The Complex Exponential and other elementary functions
CURRENT READING Brown & Curchill, pages 65-75
NEXT READING Brown & Curchill pages 75-81

Now that we know something about analytic functions in general we need to expand our repertoire of complex functions.

The Complex Exponential e^z
The complex version of the exponential function is defined like this:
$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, where $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$
$\arg(e^z) = y + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$
Exercise
Show that $f(z) = e^z$ is an *entire function* and that $f'(z) = e^z$
Take some time (3 minutes) to try and prove this. You will have to answer the questions:

1: What is an entire function?

2: How do you show that a function is analytic?

3: Do the real and complex parts of e^z obey the CRE?

More Properties of e^z

- e^z is never zero
- $e^z = 1 \iff z = 2\pi k i$
- $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i, \qquad$ where $k \in Z$
- e^z is a periodic function with period $2\pi i$

Sketch a *fundamental region* for e^z below

Other elementary functions

Once we have a handle on $\exp z$ we can use it to define other functions, most notably $\sin z$ and $\cos z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

<u>GROUPWORK</u>

Show that $\frac{d}{dz} \cos z = -\sin z$ by using the definition of $\cos z$

There a whole bunch of typical trigonometric identities which are valid for complex trig functions. Most of these can be proved using the definitions involving exponentials. For example, $\tan z$, $\sec z$ are analytic everywhere except at the zeroes of $\cos z$.

<u>GROUPWORK</u>

Find the zeroes of $\cos z$ and $\sin z$

The usual rules of derivatives of the trig functions remain valid for their complex counterparts.

Complex Trigonometric Identities

$$\begin{aligned}
\sin(z+2\pi) &= \sin z, & \cos(z+2\pi) &= \cos z \\
\sin(-z) &= -\sin z, & \cos(-z) &= \cos z \\
\sin^2 z + \cos^2 z &= 1, & \tan^2 z + 1 &= \sec^2 z \\
\sin 2z &= 2 \sin z \cos z, & \cos 2z &= \cos^2 - \sin^2 z \\
\sec z &= \frac{1}{\cos z}, & \tan z &= \frac{\sin z}{\cos z}
\end{aligned}$$

$$\frac{d}{dz} \tan z = \sec^2 z, \frac{d}{dz} \sec z = \sec z \tan z \qquad \frac{d}{dz} \sin z = \cos z, \frac{d}{dz} \sin z = \cos z$$

Similarly the hyperbolic trigonometric functions can be defined using the complex exponential and the newly-defined complex trig functions

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

Complex Hyperbolic Trigonometric Identities

$$\begin{aligned}
\sinh z &= -i \sin iz, & \cosh z &= \cos(iz) \\
\frac{d}{dz}\sinh z &= \cosh z, & \frac{d}{dz}\cosh z &= \sinh z
\end{aligned}$$