Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class 9 **(Monday February 2)**

SUMMARY The Complex Derivative
CURRENT READING Brown & Curchill, pages 45-57
NEXT READING Brown & Curchill, pages 48-50, 55-57, 59-64

Update on *Class #8*
I have analyzed your comments on the Classroom Assessment Forms, and the results are: 2 mention point sets, 2 mention complex roots, 7 mention mappings, 4 mention limits and continuity (and 3 others mention no specific topic).

Mapping
Today we will begin the class with addressing the mappings difficulties by looking at the Joukowsky Mapping. The main thing to remember when thinking about the impact of $w = f(z)$ on a complex set of points is that it involves a transformation of variables, from $z = x + iy$ into $w = u + iv$, and that the string which ties together these two objects (i.e. the z-plane and the w-plane) is the mapping function itself $f(z) = u(x, y) + iv(x, y)$.

Limits and Continuity
You need to remember the definition of what a limit is (I think it is best to consult the mental image of the coordinated shrinking of neighborhoods around z_0 and w_0) and how it relates to continuity. Here, also, your mathematical intuition derived from your experience with real functions should serve you well.

Example
Determine the image of the circle of radius r, $\quad (r \neq 1)$ under the mapping $J(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$.

Example

Let $f(z) =$ Arg (z), show that $\lim_{z\to -2}$ Arg z does not exist.

Let $f(z) = \dfrac{x^2+x}{x+y} + i\dfrac{y^2+y}{x+y}$. Compute $\lim_{z\to 0} f(z)$.

Analyticity

If the derivative $f'(z)$ exists at all points z of an *open set* G, then f is said to be **analytic** (or holomorphic or regular) on the set G.
If $f(z)$ is analytic on the whole complex plane, it is called **entire**.
If "f is analytic at the point z_0", what this really means is that f is analytic in a neigborhood of z_0. [Since "singleton" sets are closed, and not open.]
When does the derivative of a function $f(z)$ exist? What if it is written in its component form of $u(x, y)$ and $v(x, y)$? Analyticity let's us answer these questions.

Analytic functions treat the variable z as a whole unit, so that when you are given two component parts $u(x, y)$ and $v(x, y)$ they can always be combined to form a complex function of the single variable $z = x + iy$.
Consider $f_1 = x^2 - y^2 + 2xyi$ and $f_2 = x^2 - y^2 + 3xyi$
By now you should be able to see that $f_1 = z^2$ while $f_2 = z^2 + i\text{Re}(z)\text{Im}(z)$
$f_1' = 2z$ while there is no derivative of f_2

Cauchy-Riemann Equations

Analyticity implies a relationship between the real ($u(x, y)$) and imaginary ($v(x, y)$) parts of a complex function $f(z)$. That relationship is known as the **Cauchy-Riemann Equations**, which we will abbreviate C.R.E.:

$$u_x = v_y, \qquad u_y = -v_x$$

Satisfying the CRE is considered to be a *necessary* condition for analyticity of a function, because if $f(z)$ is analytic then it is necessary that the CRE are also satisfied.

$$\text{ANALYTICITY} \Rightarrow \text{C.R.E.}$$

To make satisfying the CRE a *sufficient* condition one needs the added condition that the first derivatives of u and v are continuous. If both these conditions are true and f is defined on an open set, then f is analytic on the open set.

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$$

$$\text{ANALYTICITY} \Longleftrightarrow \text{C.R.E. + Continuity of } u_x, u_y, v_x, v_y$$