Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am

Buckmire — Fowler 112

Class 8 **(Friday January 30)**

SUMMARY Continuity and Differentiability of Complex Functions
CURRENT READING Brown & Curchill, pages 40-45
NEXT READING Brown & Curchill, pages 45-57

Continuity

A complex function $f(z)$ is **continuous** at a point z_0 if *all three* of the following statements are true

1: $\lim_{z \to z_0} f(z)$ exists

2: $f(z_0)$ exists

3: $\lim_{z \to z_0} f(z) = f(z_0)$

Consider the function below:

$$f(z) = \begin{cases} \dfrac{z^2+4}{z-2i}, & z \neq 2i \\ 3+4i, & z = 2i \end{cases}$$

Answer the following questions

(a) What is the value of $\lim_{z \to 2i} f(z)$?

(b) Is $f(z)$ continuous at $z = 2i$?

(c) Is $f(z)$ continuous at points $z \neq 2i$?

We say that the function $f(z)$ defined above has a **removable singularity** at $z = 2i$. Write down the definition of $f(z)$ which has had the singularity removed.

More Aspects of Continuity

As with real functions of a real variable, **sums**, **differences**, **products** and **compositions** of continuous functions are continuous.

When $f(z)$ continuous $\Longleftrightarrow u(x,y)$ and $v(x,y)$ continuous

When $f(z)$ continouus in a region R, then $|f(z)|$ is also continuous in the region R and if R is a *bounded* and *closed* set then there exists a positive number M so that $|f(z)| \leq M \forall z \ni R$.

Derivative

Let f be defined in a neigborhood around z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$, is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the above limit exists. The function f is said to be *differentiable* at z_0.
Consider $f(z) = z^2$. Write down the expression $\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z}$

The derivative $\dfrac{dw}{dz} = f'(z)$ is defined as $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{\Delta w}{\Delta z}$
Evaluate this limit for our function $f(z) = z^2$.

Write down $f'(z)$

Write down the real and imaginary parts of the function $f(z) = z^2$

Write down the real and imaginary parts of the function $f'(z)$ See any patterns?

Rules of Differentiation

The standard rules of differentiating function that you learned for real functions basically apply to complex functions. Scilicet:

$$\frac{d}{dz}(c) = 0 \qquad \frac{d}{dz}(z) = 1 \qquad \frac{d}{dz}(z^n) = nz^{n-1} \qquad \frac{d}{dz}(e^z) = e^z$$

Linearity

$$\frac{d}{dz}[cf(z) + g(z)] = cf'(z) + g'(z) \qquad c \text{ constant}$$

Product Rule

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$$

Quotient Rule

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

Aspects of Differentiation

One of the most important aspects to remember about differentiability and continuity is:

DIFFERENTIABILITY $\Rightarrow$ CONTINUITY

CONTINUITY DOES NOT IMPLY DIFFERENTIABILITY.