Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class #7 (Wednesday January 28)

SUMMARY Limits of Complex Functions and the Point At Infinity
CURRENT READING Brown & Curchill, pages 33-40
NEXT READING Brown & Curchill, pages 40-45

Update on *Class #5*
Yes, closed sets can be connected. Open connected sets are more interesting because they are also called **domains** or open regions. If a set is closed and connected it's called a closed region.

If a set does not have any limit points, such as the set consisting of the point $\{0\}$, then it is **closed**. [It contains all its limit points (it just doesn't have any limit points).]

Remember, if a set contains all its boundary points (marked by solid line), it is **closed.**
If a set contains none of its boundary points (marked by dashed line), it is **open.**

Also, *some sets can be both open and closed.* An example is the set $\mathcal{C}$ (the Complex Plane). It has no boundary points. Thus $\mathcal{C}$ is closed since it contains all of its boundary points (doesn't have any) and $\mathcal{C}$ is open since it doesn't contain any of its boundary points (doesn't have any).

Also, *some sets can be neither open or closed.* The set $0 < |z| \leq 1$ has two boundaries (the set $|z| = 1$ and the point $z = 0$). It contains the first boundary ($|z| = 1$), so it is not open, but it does not contain the boundary point $z = 0$ so it is not closed. $z = 0$ is also a limit point for this set which is not in the set, so this is another reason the set is not closed.

Limits

Suppose that $f(z)$ is defined on a deleted neighborhood of z_0. In order to say that $\lim_{z \to z_0} f(z) = w_0$ we must be able to show that

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

This may look like dense mathematical language, but in english this means that for every positive number ϵ (no matter how small) there exists a number δ (which depends on the choice of ϵ) so that regardless of how close you get to the point z_0 in the deleted neighborhood around it in the z-plane you can also get arbitrarily close to the value w_0 in the w-plane.

Rules on Limits

The rules on limits of complex functions are identical to the rules for limits of real functions of real variables (as you'd expect)

Suppose that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$ then

$$\begin{aligned}
\lim_{z\to z_0}[f(z)+F(z)] &= w_0 + W_0 \\
\lim_{z\to z_0}[f(z)F(z)] &= w_0 W_0 \\
\lim_{z\to z_0}\frac{f(z)}{F(z)} &= \frac{w_0}{W_0} \qquad (W_0 \neq 0) \\
\lim_{z\to z_0}|f(z)| &= |w_0| \\
\lim_{z\to z_0} c &= c \\
\lim_{z\to z_0} z^n &= z_0^n \\
\lim_{z\to z_0} P(z) &= P(z_0) \qquad (\text{where } P(z) \text{ is a polynomial})
\end{aligned}$$

IF $f(z) = u(x,y) + iv(x,y), \qquad z_0 = x_0 + iy_0 \qquad$ and $w_0 = u_0 + iv_0$ THEN
$\lim_{z\to z_0} f(z) = w_0$ if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Examples

(a) $\lim_{z\to 1+2i} 2|z| + iz^2 + 2.5 - .1i =$

(b) $\lim_{z\to 3\pi i} ze^z =$

(c) $\lim_{z\to 0}\dfrac{z^8+z^4+z^2+z-1}{z^3+4z^3-9} =$

(d) $\lim_{z\to 1-i} 2xy - ix^2 - iy^2 =$

Point at Infinity

When dealing with real numbers we often speak of two different concepts, denoted $-\infty$ and $+\infty$.

However, the complex number infinity is represented as one particular point in the Argand plane. We call this the **point of infinity** and we rename the Argand plane the **extended z plane** or the **extended complex plane** when we include it.

The point at infinity can be considered to be the image of the origin $z = 0$ under the mapping $w = 1/z$

To compute complex limits involving ∞ we use this idea:

$$\lim_{z\to\infty} f(z) = \lim_{w\to 0} f\left(\frac{1}{w}\right)$$

In order for us to say a complex function $f(z)$ becomes unbounded at a point z_0, (i.e. $f(z_0) = \infty$) we must be able to show that

$$\forall M > 0, \quad \exists \delta > 0 \quad \ni 0 < |z - z_0| < \delta \Rightarrow |f(z)| > M$$

Examples

(a) $\lim_{z\to\infty}\dfrac{2z+3}{3z-i} =$

(b) $\lim_{z\to\infty} z^2 + 3z - 5i + 4 =$