Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class #4 (Wednesday 01/21/98)

SUMMARY Polynomial Equations of a Complex Variable and Roots of Complex Numbers
CURRENT READING Churchill & Brown, pages 19-23
NEXT READING Churchill & Brown, pages 23-32
Update
The answers to the exercises from the class #3 are:
1: $e^{i\pi} = \cos\pi + i\sin\pi = -1$
2: $1+i = |1+i|e^{i\,\arg(1+i)} = \sqrt{2}e^{\frac{i\pi}{4}+2ni\pi}$
3: $(1-i)^5 = \left(|1-i|e^{i\,\mathrm{Arg}\,(1-i)}\right)^5 = \left(\sqrt{2}e^{i\frac{-\pi}{4}}\right)^5 2^{5/2}e^{\frac{-i5\pi}{4}} = \sqrt{32}e^{\frac{3\pi i}{4}} = \sqrt{32}\frac{(i-1)}{\sqrt{2}} = -4+4i$

We now know how to deal with real integer powers of complex numbers in a nice way (by using DeMoivre's Formula and exponential form).

Fractional Exponents

What about real fractional powers of complex numbers, i.e. roots? That is, we want to solve an equation like

$$z^n - z_0 = 0 \tag{1}$$

where z_0 is a known complex number, and we are trying to find the corresponding value(s) of $z = z_0^{1/n}$ which solve this equation.
Suppose we write z_0, z and z^n in polar form:

$$z_0 = \tag{2}$$
$$z = \tag{3}$$
$$z^n = \tag{4}$$

where $|z_0| = r_0$, $|z| = r$, $\mathrm{Arg}\, z_0 = \theta_0$ and $\mathrm{Arg}\, z = \theta$
We can rewrite (1) as $z^n = z_0$ and obtain two real equations.

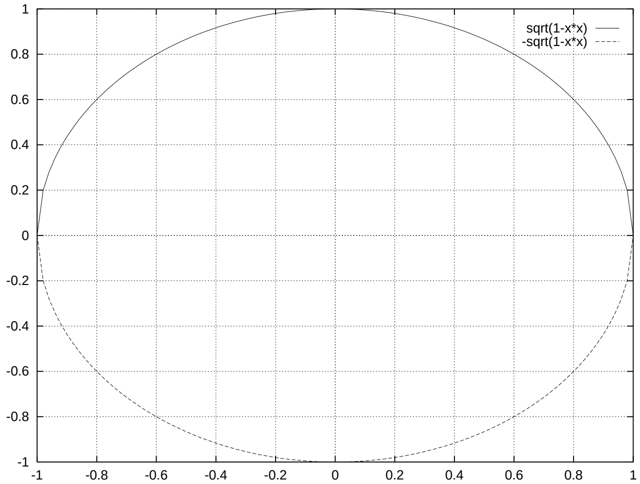
Exercise
Obtain expressions for r and θ in terms of r_0 and θ_0

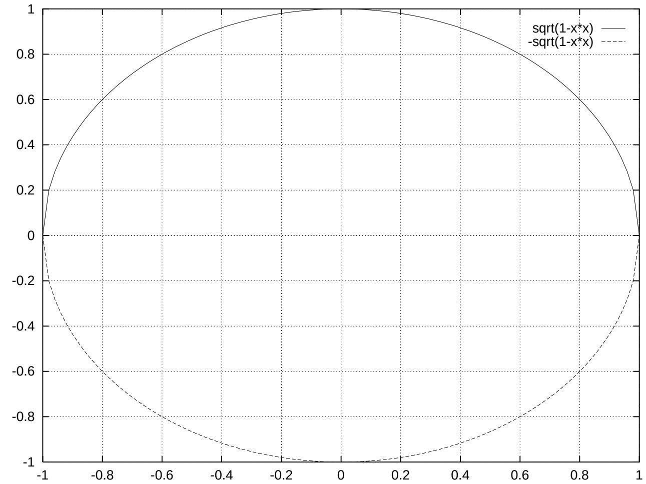
Roots of Unity

We are interested in finding the n^{th} roots of unty, i.e. z such that

$$z^n = 1, \qquad n = 1, 2, 3 \ldots$$

On the following axes, draw vector representations of the n^{th} *roots of unity* when $n = 2$, $n = 3$ or $n = 4$. How many distinct solutions to $z^n = 1$ are there? In other words, we are trying to evaluate $\sqrt{1}$, $\sqrt[3]{1}$, $\sqrt[4]{1}$, ...





What do you think the 5^{th} roots of unity will look like?

Exercise

Compute the solutions to the equation $z^5 = i$ and write them in polar (and rectangular) form. Sketch these solutions on the grid on the right.

Using DeMoivre's Formula and the result on n^{th} roots we can obtain a general formula for evaluating $z^{m/n}$

$$z^{m/n} = c_k = |z|^{m/n} \exp\left(\frac{i(\theta + 2\pi k)}{n}\right), \qquad k = 0, 1, \ldots, n-1, \quad \text{where } \theta = \text{ Arg } z$$

These n roots can be written as

$$c, c\omega_n, c\omega_n^2, c\omega_n^3, \cdots, c\omega_n^{n-1}$$

where c is *any* n^{th} root of a non-zero complex number, and $\omega_n = \exp\left(\frac{2i\pi}{n}\right)$

Examples

(1) Prove that $\sin 2\theta = 2\sin\theta\cos\theta$ and
$\cos 2\theta = \cos^2\theta - \sin^2\theta$ (use DeMoivre's Formula)

(2) Evaluate $\sqrt{5-12i}$

(3) Solve $w^3 - i = -\sqrt{3}$

(4) Solve $w^{4/3} + 2i = 0$