Class 24

TITLE Classifying Singularities and Introduction to Laurent and Maclaurin Series
CURRENT READING Zill & Shanahan, §6.6
HOMEWORK SET #10 (DUE WED APR 6)
Zill & Shanahan, §Chapter 5 Review 4,5,6,7,8,9,17. **38*,40***. §6.4 2,6,20. **25***.
§6.5 7, 2,12,17,23. §6.6 3,8,15,**24***.

SUMMARY
We shall be introduced to Laurent Series and learn how to use them to classify different various kinds of singularities (locations where complex functions are no longer analytic).

There are basically three types of singularities (points where $f(z)$ is not analytic) in the complex plane. They are called **removable singularities**, **isolated singularities** and **branch singularities**.

Isolated Singularity
An isolated singularity of a function $f(z)$ is a point z_0 such that $f(z)$ is analytic on the punctured disc $0 < |z - z_0| < r$ but is *undefined* at $z = z_0$. We usually call isolated singularities **poles**. An example is $z = i$ for the function $f(z) = \dfrac{z}{z-i}$.
Removable Singularity
A removable singularity is a point z_0 where the function $f(z_0)$ appears to be undefined but if we assign $f(z_0)$ the value w_0 with the knowledge that $\lim\limits_{z\to z_0} f(z) = w_0$ then we can say that we have "removed" the singularity. An example would be the point $z = 0$ for $f(z) = \sin(z)/z$.
Branch Singularity
A branch singularity is a point z_0 through which all possible branch cuts of a multi-valued function can be drawn to produce a single-valued function. An example of such a point would be the point $z = 0$ for $\operatorname{Log}(z)$.

Essential Singularity. The canonical example of an essential singularity is $z = 0$ for the function $f(z) = e^{1/z}$. The easiest way to define an essential singularity of a function involves Laurent Series (see the Table below reproduced from Zill & Shanahan, page 289).

$z = z_0$	**Laurent Series for** $0 < \lvert z - z_0\rvert < R$
Removable singularity	$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$
Pole of Order n	$\dfrac{a_{-n}}{(z-z_0)^n} + \dfrac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + a_0 + a_1(z-z_0) + \ldots$
Simple Pole	$\dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$
Essential Singularity	$\ldots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

Laurent series

In fact, the best way to identify an essential singularity z_0 of a function $f(z)$ (and an alternative way to compute residues) is to look at the **series representation** of the function $f(z)$ about the point z_0
That is,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \qquad R_1 < |z-z_0| < R_2$$

This formula for a Laurent series is sometimes written as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n \qquad \text{where } c_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz, \quad n = \pm 1, \pm 2, \ldots$$

This first part of this series should look somewhat familiar from your experience with real functions, since the expression is clearly a **Taylor series** if $b_n = 0$ for all n. This first part of the series representation is known as the **analytic part** of the function. The second part (with the *negative exponents*) is called the **principal part** of the function. However if a_n and b_n are not all identically zero this type of series is called a **Laurent series** and converges to the function $f(z)$ in the annular region $R_1 < |z-z_0| < R_2$.

$\boxed{\text{EXAMPLE}}$

Let's show why expressing the function $f(z)$ in terms of a Laurent Series is useful by proving that the value of the $\mathbf{Res}(f; z_0)$ is exactly equal to b_1 (or c_{-1}), that is, the coefficient of the $\dfrac{1}{z-z_0}$ term. We can do this by integrating the Laurent series term by term on some closed contour C and using the CIF.

Our goal is to see how series representations of functions allows us to compute integrals of functions easily by computing residues, and other means. But first we need to update our knowledge of sequences and series of real variables to the same objects using complex variables.

Review of Sequences and Series

Recall that an infinite **sequence** $\{z_n\}$ converges to z if for each $\epsilon > 0$ there exists an N such that if $n > N$ then $|z_n - z| < \epsilon$

The sequence $z_1, z_2, z_3, \dots, z_n, \dots$ converges to the value $z = x + iy$ if and only if the sequence $x_1, x_2, x_3, \dots$ converges to x and $y_1, y_2, y_3, \dots$ converges to y.

In other words $\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

An infinite **series** $\displaystyle\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$ converges to S if the sequence S_N of **partial sums** where $S_N = z_1 + z_2 + z_3 + z_4 + \cdots + z_N \qquad (N = 1, 2, 3, \dots)$ converges to S. Then we say that $\displaystyle\sum_{n=1}^{\infty} z_n = S$.

As with sequences, series can be split up into real and imaginary parts. Suppose $z_n = x_n + iy_n$ and $\displaystyle\sum_{n=1}^{\infty} z_n = Z, \sum_{n=1}^{\infty} x_n = X$ and $Y = \displaystyle\sum_{n=1}^{\infty} y_n$ then $Z = X + iY$.

Taylor series

Suppose a function f is analytic throughout an open disk $|z - z_0| < R_0$ centered at z_0 with radius R_0. Then at each point z in this disk $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!} \text{ for } (n = 0, 1, 2, \dots)$$

In other words the function $f(z)$ can be represented exactly by the infinite series in the disk $|z - z_0| < R$

When $z_0 = 0$ the series is known as a **Maclaurin series**.

Here are some examples of well known Maclaurin series you should know.

$$\begin{array}{lclcll}
e^z & = & 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots & = & \sum_{k=0}^{\infty} \frac{z^k}{k!} & |z| < \infty \\
\sin(z) & = & z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots & = & \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k+1}}{(2k+1)!} & |z| < \infty \\
\cos(z) & = & 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots & = & \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k)!} & |z| < \infty \\
\sinh(z) & = & z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots & = & \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} & |z| < \infty \\
\cosh(z) & = & 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots & = & \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} & |z| < \infty \\
\frac{1}{1-z} & = & 1 + z + z^2 + z^3 + \dots & = & \sum_{k=0}^{\infty} z^k & |z| < 1 \\
\ln(1+z) & = & z - \frac{z^2}{2} + \frac{z^3}{3} + \dots & = & \sum_{k=1}^{\infty} \frac{z^k}{k} & |z| < 1 \\
(1+z)^p & = & 1 + pz + \frac{p(p-1)z^2}{2!} + \dots \frac{p(p-1)\dots(p-n+1)}{n!} \dots & & & |z| < 1 \\
\tan(z) & = & z + \frac{z^3}{3} + \frac{2z^5}{15} + \dots & & & |z| < \frac{\pi}{2}
\end{array}$$

$\boxed{\text{EXAMPLE}}$

1. Consider the function $f(z) = \dfrac{\sin(z)}{z^4}$. Write down the Laurent Series for this function and use this expansion to obtain $\mathbf{Res}(f;0)$. Classify the singularity at $z = 0$.

2. Confirm your value of $\mathbf{Res}\left(\dfrac{\sin z}{z^4};0\right)$ by direct computation (use the Residue formula).

3. Evaluate $\displaystyle\oint_{|z|=2} \frac{\sin(z)}{z^4}\,dz$

$\boxed{\text{GROUPWORK}}$

1. Write down the Laurent series for $f(z) = e^{1/z}$ in the region $0 < |z| < \infty$.

2. What is the value of $\mathrm{Res}(e^{1/z}, 0)$?

3. Classify the singularity at $z = 0$.

4. Evaluate $\displaystyle\oint_{|z|=2} e^{1/z}\,dz$