# Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

(BY:) 2016 Ron Buckmire — http://sites.oxy.edu/ron/math/312/16/

## *Class 12*: Wednesday February 17

**TITLE** Applications of Harmonic Functions
**CURRENT READING** Zill & Shanahan, Section 3.5.
**HOMEWORK SET #5 (DUE WED FEB 24)**
Zill & Shanahan, §3.2 #6,8,37. **19***; §3.3 #1, 3, 15, 18, 24. **27***;
Zill & Shanahan, §3.4 #6, 11, **14***; §3.5 #7, 12, **16***; Chapter 3 Review : 6,7,8,9,10, **27***;

**SUMMARY**
We shall begin looking at applications of analytic functions by taking a closer look at harmonic functions and situations in which they can be useful.

## The Complex Velocity Potential

In Fluid Dynamics, the complex velocity potential is a useful quantity used to analyze certain fluid fields. It can be defined as

$$\Phi(z) = \phi(x, y) + i\psi(x, y)$$

where $\phi(x, y)$ is called the *velocity potential* and $\psi(x, y)$ is called the *stream function*. Potential functions are useful because simply by taking the proper partial derivative one can obtain the velocity components.

One can compute expressions for $V_x$ (horizontal component) and $V_y$ (vertical component) of the velocity of a fluid, denoted by $\vec{v}$ from the complex velocity potential:

$$\overline{\frac{\partial \Phi}{\partial z}} = V_x + iV_y = \frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$$

Note that $\vec{v} = \begin{bmatrix} V_x \\ V_y \end{bmatrix}$

**Exercise**

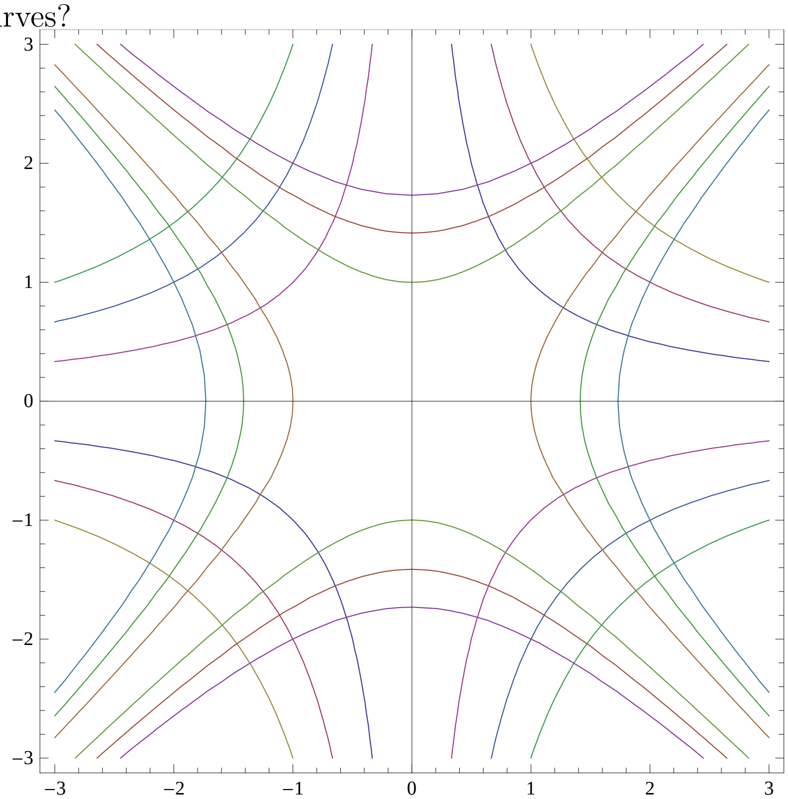
Given $\Phi(z) = z^2$ compute the stream function $\psi$ and velocity potential $\phi$ and obtain expressions for fluid velocity $\vec{v}(x, y)$

## Streamlines and Equipotentials

Recall that the level curves of a function $f(x, y)$ occur when $f(x, y) =$ constant. Level curves of $\phi(x, y)$ are called **equipotentials** and level curves of $\psi(x, y)$ are called **streamlines**. They have particularly interesting physical meaning.

**GroupWork**

Observe the streamlines and equipotentials relating to the flow described by $\Phi(z) = z^2$ in the figure below. In other words, these curves represent $\phi(x, y) = c$ and $\psi(x, y) = d$, where $c$ and $d$ are constants such as $\pm 1, \pm 2$, et cetera. Identify which curves are the streamlines and which are the equipotentials. EXPLAIN YOUR ANSWER. What is the name given to these kinds of curves?




If you look carefully the equipotentials and streamlines intersect at right angles. This is not an accident. Level curves for the real and imaginary parts of an analytic function $f(z)$ are always **orthogonal**. Pretty cool, huh?

We can show this by remembering the meaning of the gradient of a function $f(x, y)$, denoted by $\nabla f$, the dot product and applying the Cauchy-Riemann equations:

EXAMPLE

Show that the level curves of harmonic conjugates of an analytic function always intersect perpendicularly.

## The Dirichlet Problem

A problem where one is looking for a function $\phi(x, y)$ which satisfies a partial differential equation (like Laplace's Equation) in an open connected set $D$ (i.e. a domain) and which equals equals a known function $g(x, y)$ along the boundary of $D$ (sometimes represented by $\partial D$) is called a **Dirichlet problem**.

**Exercise**

Given the following Dirichlet problem for Laplace's Equation

$$\phi_{xx} + \phi_{yy} = 0 \qquad x_0 < x < x_1, \qquad -\infty < y < \infty$$

$$\phi(x_0, y) = k_0, \quad \phi(x_1, y) = k_1, \qquad -\infty < y < \infty$$

show that the function

$$\phi(x, y) = \frac{k_1 - k_0}{x_1 - x_0}(x - x_0) + k_0$$

is a solution of the given Dirichlet problem.

EXAMPLE

We can find the complex velocity potential $\Phi$ by obtaining and expression for the stream function $\psi$, the harmonic conjugate of $\phi$.

**GroupWork**

Adapted from **Zill & Shanahan, page 146, #11**. Let $D = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$. Given the boundary conditions are $\phi(0, y) = 50$ and $\phi(1, y) = 0$.

**(a)** Find the solution $\phi(x, y)$ to the corresponding Dirichlet problem on $D$ for Laplace's equation.

**(b)** Find the corresponding complex potential $\Phi(z)$.

**(c)** Sketch representative streamlines and equipotentials of the flow corresponding to $\Phi$.

EXAMPLE

Adapted from **Zill & Shanahan, page 146, #9**. Let $\phi(x, y) = x + \dfrac{x}{x^2 + y^2}$.

**(a)** Show that the corresponding complex velocity potential $\Phi$ is given by $\Phi(z) = z + \dfrac{1}{z}$.

**(b)** Compute an expression for the corresponding velocity field $\vec{v}$.