Complex Analysis

Math 312 Spring 2016 Fowler 309 MWF 11:45am-12:40pm

 http://sites.oxy.edu/ron/math/312/16/

Class 11: **Friday February 12**

TITLE Analyticity, the Cauchy-Riemann Equations and Harmonic Functions
CURRENT READING Zill & Shanahan, Section 3.3 and 3.4.
HOMEWORK SET #4 (DUE WED FEB 17)
Zill & Shanahan, Chap 2 Review 1-10, §3.1.1: #2, 11, 17, **20***; §3.1.2: #28, 31, 37, **50***;

SUMMARY
We shall move on from our definition of differentiability to the idea of **analyticity** and the famous Cauchy-Riemann Equations. We'll also introduce the concept of harmonic functions and the harmonic conjugate. The important application of analytic functions to mathematical physics will be noted.

Derivation of the Cauchy-Riemann Equations

We shall derive the Cauchy-Riemann equations by looking at the definition of the derivative of a function $f(z) = u(x, y) + iv(x, y)$ at the point z_0.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x + i\Delta y} \end{aligned}$$

We shall do this limit twice, once letting $\Delta z \to 0$ horizontally and the other time letting $\Delta z \to 0$ vertically

Difference Between Analyticity and Differentiability
Differentiability is a property of a function that occurs at a particular point. If a function is differentiable at every point in a set, then we can say that it is differentiable on that set. (But if that set is open, then we would also say that the function is analytic on that set.)

Remember analyticity is a property a function that is defined on an open set, often times a neighborhood of a particular point.

THEOREM

A complex function $w = f(z)$ is said to be **analytic** (or "regular" or "holomorphic") **at a point** z_0 if f is differentiable at z_0 **and at every point** in a neighborhood surrounding z_0.

DIFFERENTIABILITY ON AN OPEN SET $\iff$ ANALYTICITY ON THAT OPEN SET

The Cauchy-Riemann Equations and Analyticity

Given a function $f(z) = u(x, y) + iv(x, y)$ the corresponding Cauchy-Riemann Equations are

$$u_x = v_y, u_y = -v_x$$

$$\text{ANALYTICITY} \Rightarrow \text{C.R.E. satisfied}$$

To make satisfying the CRE a *sufficient* condition for analyticity one needs the added condition that the first derivatives of u and v are continuous on an open set. If both these conditions are true and f is defined on an open set, then f is analytic on the open set.

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$$

$$\text{ANALYTICITY} \iff \text{C.R.E.} + \text{Continuity of } u_x, u_y, v_x, v_y$$

EXAMPLE

Show that $f(z) = \overline{z}$ is **not analytic** anywhere in the complex plane. You can do this in two ways:

1:

2:

GroupWork

Show that the function $f(z) = 1/z$ is analytic on the set $\{z \in \mathbb{C} : z \neq 0\}$. To do that you will have to answer the following questions:

- What is its domain of definition? Is this an open set?
- What are its component functions? Are their partial derivatives continuous?
- Do they satisfy the CRE?
- Is it analytic? On what set? Is this set open or closed?

Laplace's Equation
The partial differential equation shown below is known as **Laplace's Equation**.

$$\nabla^2\phi = \Delta\phi = \frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0$$

Solutions $\phi(x,y)$ which solve Laplace's equation are very important in a number of areas of mathematical physics and applied mathematics. Some of these applications are:

- electrostatic potential in two-dimensional free space
- scalar magnetostatic potential
- stream function and velocity potential in fluid flow (aerodynamics, etc)
- spatial distribution of equilibrium temperature

Harmonic Functions
A real-valued function $\phi(x,y)$ is said to be **harmonic** in a domain (i.e. open, connected set) $\mathcal{D}$ if all its second-order partial derivatives are continuous in $\mathcal{D}$ and if ϕ satisfies Laplace's Equation at each point $(x,y) \in \mathcal{D}$.

THEOREM

If $f(z)$ is analytic on a domain $\mathcal{D}$ then both $u(x,y) =\text{Re}(f(z))$ and $v(x,y) =\text{Im}(f(z))$ are harmonic in $\mathcal{D}$.

$$\text{ANALYTICITY} \Longleftrightarrow \text{ Re } f(z) \text{ and } \text{ Im } f(z) \text{ are HARMONIC}$$

PROOF

The proof follows directly from the CRE.
(Take 3 minutes and try and come up with it.)

Given a harmonic function $u(x,y)$ defined on an open connected set $\mathcal{D}$ we can construct a **harmonic conjugate** $v(x,y)$ so that the combined function $f = u(x,y) + iv(x,y)$ will be analytic on the domain $\mathcal{D}$.

EXAMPLE

Given $u(x,y) = x^3 - 3xy^2 + y$ find the harmonic conjugate $v(x,y)$ and thus construct an analytic function $f(z)$ such that Re $f(z) = u(x,y)$