HW 9
MATH 312

5.3: 2, 9, 12, 20, 25, 27, 23*, 29*
5.4: 1, 8, 18, 22, 25*
5.5: 7, 22, 23, 24

5.3

12 $\int_C \left(z + \frac{1}{z^2}\right) dz = \int_C z\,dz + \int_C \frac{1}{z^2}\,dz$

$C: |z| = 2$ $\quad = 0 + 0$

0 ← CGT

0 ← because $\int \frac{dz}{(z-z_0)^n} = 0$ unless $n = 1$

27 (a) $f(z) = (6i z^4 - 4z^2 + 2 - 6i)^9$

$\oint_C f(z)\,dz = 0$ because $f(z)$ is an entire function since it is a composition of polynomials

(b) $f(z) = (z^2 - 3iz) e^{5z}$

This is also an entire function because its the product of two entire function, exponential & polynomial

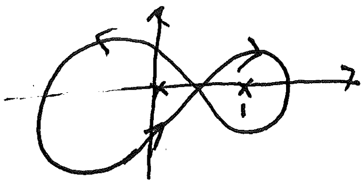
(c) $f(z) = \frac{\sin z}{e^{z^2}}$ This is also an entire function because e^z never output a zero so it's the product of 2 entire functions

(d) $f(z) = z \cos^2 z$ This is an entire function because its the product of two entire functions

MATH 312
HW 9

5.3: 2, 9, 12, 20, 25, 27, 23*, 29*

23* $\displaystyle\oint_C \frac{8z-3}{z(z-1)}\,dz = 2\pi i\left(\frac{8z-3}{z-1}\right)\Bigg|_{z=0} - 2\pi i\left(\frac{8z-3}{z}\right)\Bigg|_{z=1}$



$$= 2\pi i\left(\frac{-3}{-1}\right) - 2\pi i\left(\frac{5}{1}\right)$$

$$= 2\pi i(3-5)$$

$$= \boxed{-4\pi i}$$

29* $\displaystyle\oint_{|z|=1} \frac{e^z}{z+3} - \frac{3}{z}\,dz = \oint_{|z|=1} \frac{e^z}{z+3}\,dz - 3\oint \frac{1}{z}\,dz$

$$= 0 - 3\cdot 2\pi i = -6\pi i$$

$\displaystyle\oint \frac{e^z}{z+3} - 3\bar{z}\,dz = \oint \frac{e^z}{z+3}\,dz - 3\oint \bar{z}\,dz$

$$= 0 - 3\cdot 2i(\text{Area of circle of radius 1})$$

$$= -6\pi i$$

MATH312
HW9

5.4: 1, 8, 18, 22, 25*

18 $\displaystyle\int_C \frac{1}{z}\,dz = \operatorname{Ln} z \Big|_{1+i}^{4+4i} = \operatorname{Ln}(4+4i) - \operatorname{Ln}(1+i)$

$C: 1+i \to 4+4i$

$= \ln|4+4i| + \frac{i\pi}{4} - \left(\ln|1+i| + i\frac{\pi}{4}\right)$

$= \ln 4\sqrt{2} - \ln\sqrt{2}$

$= \ln 4$

22 $\displaystyle\int_0^i z\sin z\,dz = -\frac{i}{e}$

IBP
$u = z \qquad du = dz$
$dv = \sin z\,dz \qquad v = -\cos z$

$\displaystyle = -z\cos z\Big|_0^i - \int_0^i -\cos z\,dz$

$\displaystyle = -i\cos i - (-0\cdot\cos 0) + \int_0^i \cos z\,dz$

$= -i\cos i + \sin i = -i\left(\frac{e^{-1}+e^{1}}{2}\right) + \left(\frac{e^{-1}-e^{1}}{2i}\right) = e^{-1}\left(-\frac{i}{2}-\frac{i}{2}\right) + e^{1}\left(-\frac{i}{2}+\frac{i}{2}\right) = \boxed{-\frac{i}{e}}$

25* $\displaystyle\int_C \frac{1}{4z^{1/2}}\,dz = \frac{z^{1/2}}{2}\Bigg|_{-4i}^{+4i} = \frac{(4i)^{1/2}}{2} - \frac{(-4i)^{1/2}}{2}$

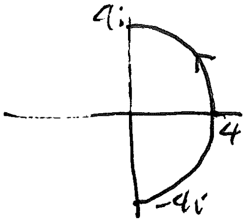
$= i^{1/2} - (-i)^{1/2}$

$= (e^{i\pi/2})^{1/2} - (e^{-i\pi/2})^{1/2}$

$= e^{i\pi/4} - e^{-i\pi/4}$

$= \frac{1+i}{\sqrt{2}} - \left(\frac{1-i}{\sqrt{2}}\right)$

$= \frac{2i}{\sqrt{2}} = \boxed{i\sqrt{2}}$ using FTCI



$z(t) = 4e^{it}$
$z' = 4ie^{it}$

$\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{4}\cdot\frac{1}{2e^{it/2}}\cdot 4ie^{it}\,dt$

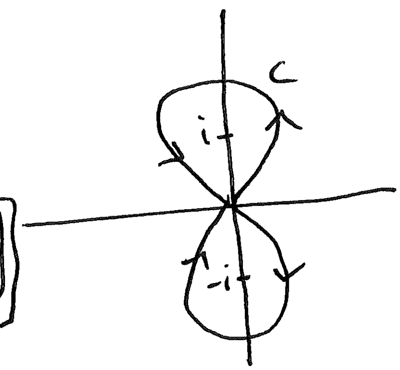
$\displaystyle\frac{i}{2}\int_{-\pi/2}^{\pi/2} e^{it/2}\,dt = e^{it/2}\Big|_{-\pi/2}^{\pi/2}$

$= e^{i\pi/4} - e^{-i\pi/4} = \boxed{i\sqrt{2}}$

MATH 312
HW9

5.5 : 7, 22, 23, 24

24. $\displaystyle\int_C \frac{e^{iz}}{(z^2+1)^2}\,dz$



$$= 2\pi i\left[+\mathrm{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, i\right) + \mathrm{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, -i\right)\right]$$

There are poles of order 2 at $z^2+1=0 \Leftrightarrow z=\pm i$

$$\mathrm{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, i\right) = R_1$$

$$R_1 = \frac{1}{1!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2\frac{e^{iz}}{(z^2+1)^2}\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{e^{iz}}{(z+i)^2}\right] = \lim_{z\to i}\left[\frac{ie^{iz}}{(z+i)^2} + e^{iz}\cdot\frac{-2}{(z+i)^3}\cdot 1\right]$$

$$= \frac{ie^{-1}}{(2i)^2} + \frac{e^{-1}(-2)}{(2i)^3} = \frac{ie^{-1}}{-4} + \frac{-2e^{-1}}{-8i} = \frac{1}{4e}(-i-i) = \boxed{\frac{-i}{2e}}$$

$$R_2 = \frac{1}{1!}\lim_{z\to -i}\frac{d}{dz}\left[(z+i)^2\frac{e^{iz}}{(z^2+1)^2}\right] = \lim_{z\to -i}\frac{d}{dz}\left[\frac{e^{iz}}{(z-i)^2}\right]$$

$$= \lim_{z\to -i} ie^{iz}\cdot\frac{1}{(z-i)^2} + e^{iz}\cdot\frac{-2}{(z-i)^3} = ie\,\frac{1}{(-2i)^2} + e\cdot\frac{-2}{(-2i)^3}$$

$$= \frac{ie}{-4} + e\frac{(-2)}{-8\cdot(-i)} = -\frac{ie}{4} + \frac{e}{4}\frac{1}{(-i)} = -\frac{ie}{4} + i\frac{e}{4} = 0$$

$$\int_C \frac{e^{iz}}{(z^2+1)^2}\,dz = 2\pi i\left[-0 + -\frac{i}{2e}\right] = 2\pi\cancel{i}\left(-\frac{i}{2e}\right) = \boxed{\frac{\pi}{e}}$$