MATH 312
HW Set 5

3.2 6, 8, 37, 19*
3.3 1, 3, 15, 18, 24, 27*
3.4 6, 11, 19*
3.5 7, 12, 16*
Chap 3 6, 7, 8, 9, 10, 27*

Section 3.2

6. Use Limit definition to find derivative of $f(z) = \frac{-1}{z^2}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{-1}{(z+\Delta z)^2} - \frac{-1}{z^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 - (z^2 + 2z\Delta z + (\Delta z)^2)}{z^2 \cdot (z+\Delta z)^2 \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-2z - \Delta z}{z^2 \cdot (z+\Delta z)^2} = \frac{-2z}{z^4} = -\frac{2}{z^3}$$

8. $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \to z_0} \frac{(z - z_0)(z^2 + zz_0 + z_0^2)}{(z - z_0)}$$

$$= \lim_{z \to z_0} z^2 + zz_0 + z_0^2 = z_0^2 + z_0^2 + z_0^2 = 3z_0^2$$

$$f'(z_0) = 3z_0^2$$

37. Suppose $f'(z)$ exists at z, is $f'(z)$ continuous at z?
No! $f(z)$ is continuous at z but we have no info about $f'(z)$ at z

19.* $f(z) = |z|^2 = x^2 + y^2 = u + iv$

$u_x = 2x = v_y = 0$

$u_y = 2y = -v_x = 0$ $\Rightarrow$ CREs only true when $x = 0$ & $y = 0$

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{|z|^2}{z} = \lim_{z \to 0} \frac{z\bar{z}}{z} = \lim_{z \to 0} \bar{z} = 0$$

$f'(z)$ exists at $z = 0$.
CREs not true $\Rightarrow f'(z)$ not existing when $z \neq 0$

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3.3 : 1, 3, 15, 18, 24, 27*

1. $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$
$= x^3 - 3xy^2 + i(3x^2y - y^3)$

$u = x^3 - 3xy^2$ $\quad u_x = 3x^2 - 3y^2 \quad u_y = -6xy$

$v = 3x^2y - y^3$ $\quad v_x = 6xy \quad v_y = 3x^2 - 3y^2$

$u_x = v_y$ ✓

$u_y = -v_x$ ✓

3. $f(z) = \text{Re}(z)$

$u = x \quad u_x = 1 \quad u_y = 0 \quad u_x \neq v_y$

$v = 0 \quad v_x = 0 \quad v_y = 0 \quad u_y = -v_x$ ✓

CREs not satisfied for any value means $f'(z)$ does not exist anywhere.

15. $f(z) = \dfrac{\cos\theta}{r} - i\dfrac{\sin\theta}{r}$

CREs in polar

$u_r = \frac{1}{r} v_\theta$

$v_r = -\frac{1}{r} u_\theta$

$u = \dfrac{\cos\theta}{r} \quad v = -\dfrac{\sin\theta}{r}$

$u_r = -\dfrac{\cos\theta}{r^2} \quad u_\theta = \dfrac{-\sin\theta}{r}$

$v_r = +\dfrac{1}{r^2}\sin\theta \quad v_\theta = -\dfrac{\cos\theta}{r}$

$-\dfrac{\cos\theta}{r^2} \overset{?}{=} \dfrac{1}{r}\left(-\dfrac{\cos\theta}{r}\right)$ ✓

$\dfrac{1}{r^2}\sin\theta = -\dfrac{1}{r}\left(\dfrac{-\sin\theta}{r}\right)$

✓ $= \dfrac{\sin\theta}{r^2}$

$f'(z) = e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}\left(-\dfrac{\cos\theta}{r^2} - i\dfrac{\sin\theta}{r^2}\right)$

$= -\dfrac{1}{r^2} e^{-i\theta} e^{i\theta} = -\dfrac{1}{r^2}$

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(18) $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$

$u_x = 2x + ay \qquad v_x = 2cx + dy$

$u_y = ax + 2by \qquad v_y = dx + 2y$

$u_x = v_y \Rightarrow 2x + ay = dx + 2y$

$a = 2$

$2 = d$

$u_y = -v_x \Rightarrow ax + 2by = \cancel{-dx - 2y}\ -(2cx + dy)$

$ax + 2by = \cancel{-dx}$

$2x = -2c \Rightarrow c = -1$

$2b = -d \qquad b = -1$

$\cancel{a = d}$

$\cancel{b = 2}$

$f(z) = x^2 + 2xy - y^2 + i(-x^2 + 2xy + y^2)$

(24) $f(z) = x^2 - x + y + i(y^2 - 5y - x)$ $\qquad u_x = 2x - 1 \qquad u_y = 1$

$u = x^2 - x + y \qquad v_x = -1 \qquad v_y = 2y - 5$

$v = y^2 - 5y + x$

$u_x = v_y$

$2x - 1 = 2y - 5 \Rightarrow 2y = 2x + 4 \Rightarrow y = x + 2$

$u_y = -v_x$

$1 = -(-1)$ ✓

CREs only true when $y = x + 2$

This mean $f'(z)$ only exists along this path

& $f(z)$ is NOWHERE analytic

$f'(z) = u_x + iv_x = 2x - 1 - i$

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27*. $f'(z) = u_x + iv_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2}$

$|f'(z)|^2 = u_x^2 + v_x^2 = (v_y)^2 + (-u_y)^2$

$= v_y^2 + u_y^2$

3.4 6, 11, 14*

6 $u(x,y) = \cos x \cosh y$

$u_y = \cos x \sinh y = -v_x \Rightarrow v = -\sin x \sinh y + \phi(y)$

$u_x = -\sin x \cosh y = +v_y = -(\sin x \cosh y) - \phi'(y)$

$-\phi'(y) = 0 \Rightarrow \phi(y) = c$

$f = \cos x \cosh y \cancel{+} i \sin x \sinh y$

$= \left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) \cancel{+} i\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{y} - e^{-y}}{2}\right)$

$= \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

$= \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2} = \frac{e^{ix}e^{y} + e^{ix}e^{-y} + \cancel{e^{-ix+y}} + \cancel{e^{-ix-y}}}{4} + \frac{e^{ix}e^{y} + e^{-ix-y} - \cancel{e^{-y}} - \cancel{e^{ix}}}{4}$

11

$\boxed{u(x,y) = xy + x + 2y - 5}$

$u_x = y + 1 = v_y \Rightarrow v = \frac{y^2}{2} + y + \phi(x)$

$u_y = x + 2$

$v_x = \phi'(x) = x + 2$

$\phi = \frac{x^2}{2} + 2x + C$

$v = \frac{y^2}{2} + \frac{x^2}{2} + y + 2x + C$

$f(2i) = -1 + 5i$ $\quad x = 0, y = 2 \Rightarrow u = -1, v = 5$

$\frac{2^2}{2} + \frac{0^2}{2} + 2 + 0 + C = 5$

$4 + C = 5$

$C = 1$

$\boxed{v = \frac{y^2}{2} + \frac{x^2}{2} + y + 2x + 1}$

$f(z) = u + iv = \cancel{\frac{z^2}{2} + z - 2i + z}$

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3.4 #14* $r u_r = v_\theta \quad r v_r = -u_\theta$

$$(r u_r)_r = (v_\theta)_r \qquad (r v_r)_\theta = (-u_\theta)_\theta$$

$$(r u_r)_r = v_{\theta r} \qquad \Rightarrow r v_{r\theta} = -u_{\theta\theta}$$

$$v_{r\theta} = -\frac{1}{r} u_{\theta\theta}$$

$$(r u_r)_r = -\frac{1}{r} u_{\theta\theta}$$

$$r u_{rr} + u_r + \frac{1}{r} u_{\theta\theta} = 0$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \ \square$$

This usually written as

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \ \square \quad \text{(Laplace's Eq}^n\text{)}$$

3.5 : 7, 12, 16*

7. $f(z) = e^{-x}\cos y - e^{-x}\sin y$

$u = e^{-x}\cos y = c$

$v = -e^{-x}\sin y$

$$\frac{dy}{dx} u_y + u_x = 0$$

$$\frac{dy}{dx} \cdot (-e^{-x}\sin y) + (-e^{-x}\cos y) = 0$$

$$\frac{dy}{dx} = \frac{e^{-x}\cos y}{-e^{-x}\sin y} = -\cot y$$

$$\frac{dy}{dx} \cdot v_y + v_x = 0$$

$$\frac{dy}{dx} = -\frac{v_x}{v_y} = -\frac{e^{-x}\sin y}{-e^{-x}\cos y} = \tan y$$

Product of $\left.\frac{dy}{dx}\right|_{u=c} \cdot \left.\frac{dy}{dx}\right|_{v=c} = -\frac{1}{\tan y} \cdot \tan y = -1$

proves orthogonal

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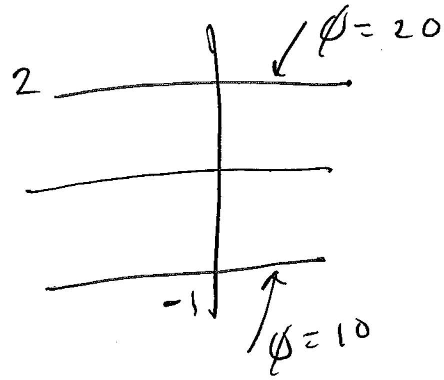


3.5.12 $\phi_{xx} + \phi_{yy} = 0$

$\phi(x,-1) = 10$

$\phi(x,2) = 20$

$$\phi(x,y) = \frac{K_1 - K_0}{y_1 - y_0}(y - y_0) + K_0$$

$y_0 = -1 \quad K_0 = 10$

$y_1 = 2 \quad K_1 = 20$

$$\phi(x,y) = \frac{20-10}{2-(-1)}(y-(-1)) + 10$$

$$\phi(x,y) = \frac{10}{3}(y+1) + 10$$

$\phi_x = \psi_y$

$0 = \psi_y \Rightarrow \psi = C + A(x)$

$= -\phi_y = \psi_x = A'(x)$

$-\left(\frac{10}{3}\right)$

$-\frac{10}{3} = A'(x)$

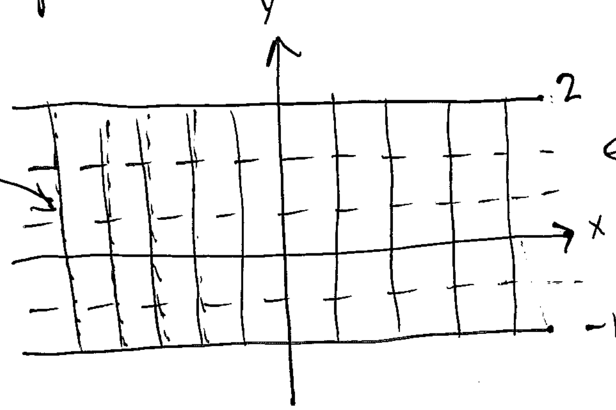
$-\frac{10}{3}x + B = A(x) \Rightarrow \psi = C + B - \frac{10x}{3}$

$$\Omega(z) = \phi + i\psi = \frac{10}{3}(y+1) + 10 + i\left(-\frac{10}{3}x + C + B\right)$$

$$\Omega(z) = -\frac{10}{3}iz + \frac{40}{3} + i(C+B)$$

Equipotential lines occur where $\phi = c$

lines of force are VERTICAL



equipotentials are HORIZONTAL

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Chapter 3 REVIEW
6, ~~7~~, 8, 9, 10, 27*

6. "If f is differentiable at a point z, then f is analytic at z!"

$\boxed{\text{FALSE!}}$ You can be DIFFERENTIABLE at a point without being ANALYTIC there.
ANALYTICITY requires the derivative to exist at a point AND EVERY POINT OF SOME NEIGHBORHOOD around that point

7. "$f(z) = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2}$ is differentiable for all $z \neq 0$"

$\boxed{\text{TRUE}}$

$$u = \frac{y}{x^2+y^2}, \quad v = \frac{x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2} \qquad u_y = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$v_x = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} \qquad v_y = \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}$$

$$u_x = \frac{-2xy}{(x^2+y^2)^2} \stackrel{?}{=} -\frac{2xy}{(x^2+y^2)} = v_y \quad \text{YES!}$$

$$u_y = \frac{x^2-y^2}{(x^2+y^2)^2} \stackrel{?}{=} -\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) = -v_x \quad \text{YES!}$$

Since CREs are ~~abot~~ satisfied and u_x, u_y, v_x, v_y are continuous EVERYWHERE except $x = y = 0$, this $f'(z)$ exists $\forall z, z \neq 0$.

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Chap 3 Review. 8

"The function $f(z) = z^2 + \bar{z}$ is nowhere analytic."

TRUE. $f(z) = \bar{z}$ is nowhere analytic, so even if you add an analytic function (z^2) to it, it won't be analytic

9 "The function $f(z) = \cos y - i \sin y$ is nowhere differentiable."

$u = \cos y \quad v = -\sin y$

$u_x = 0 \quad v_x = 0$

$u_y = -\sin y \quad v_y = -\cos y$

CRES $\quad u_x = v_y$

$0 = -\cos y \Rightarrow y = \pi/2 \pm k\pi,\ k \in \mathbb{Z}$

$u_y = -v_x \Rightarrow y = k\pi,\ k \in \mathbb{Z}$

$-\sin y = 0$

TRUE There is no y value where CRES satisfied

10 "There does not exist an analytic function $f = u + iv$ so that $u = y^3 + 5x$."

$u = y^3 + 5x$

$u_x = 5 = v_y \Rightarrow v = 5y + \phi(x)$

$u_y = 3y^2 = -v_x = -\phi'(x) \Rightarrow \phi'(x) = -3y^2$

$\phi(x) = -3y^2 x + c$

$v = 5y + 3y^2 x + c$

$f = y^3 + 5x + i(5y - 3y^2 x + c)$ will be analytic since CRES are satisfied for all (x, y) & u_x, u_y, v_x, v_y continuous

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Chapter 2.7* .13

$$f(z) = 2x^3 + 3iy^2 = u + iv$$

$$f'(z) = 6x^2 + i\,0 = u_x + i v_x$$

$$f'(x + ix^2) = 6x^2$$