Test 2: Complex Analysis

Math 312 Spring 2016 April 15, 2016
©2016 Ron Buckmire **11:45am-12:40pm**

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Directions:

Read *all* problems first before answering any of them. This tests consists of four (4) problems (and a BONUS problem) on seven (7) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, limited-notes*, closed book, test. **No calculators or electronic devices may be used.**

There is to be no communication during this test with any other person (except the proctor). Your work must be your own.

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

***You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam when you hand it in.**

FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!

Pledge: I, ____, pledge my honor as a human being and a member of the Occidental College community, that I will follow the above rules. I also pledge that I will not lie, cheat or steal and that I will report any such violation that I may witness.

No.	Score	Maximum
1		30
2		20
3		25
4		25
EXTRA CREDIT		5
Total		**100**

1. *[30 points.]* VERBAL, ANALYTIC. **Complex Exponential, Complex Logarithm, Complex Powers, Cauchy Integral Theorems, .** Consider the following statements and fill in the box with either TRUE or FALSE for each of the five statements below in this question.

To be true, the statement must ALWAYS be true. If you think the statement is FALSE, provide a counter-example which disproves the statement. If you think the statement is TRUE, you should provide (correct!) reasoning which proves the statement is true. You will receive *1 point* for your choice of TRUE/FALSE and *4 points* for your explanation or counterexample.

(a) *[5 pts.]* **TRUE or FALSE?** "The expression i^π is represented graphically as an infinite number of points lying somewhere on the unit circle $|z| = 1$."

$\boxed{\text{TRUE}}$

$$i^\pi = e^{\log(i^\pi)} = e^{\pi \log i} = e^{\pi(\ln 1 + i \arg i)}$$

$$= e^{\pi i \arg i} = e^{i \pi \arg i}$$

$$= e^{i\pi \cdot \left(\frac{\pi}{2} + 2k\pi\right)}, \quad k \in \mathbb{Z}$$

This will look like an infinite number of points on $|z| = 1$

$$|i^\pi| = \left|e^{i \frac{\pi^2}{2}\left(\frac{1}{2} + 2k\right)}\right| = 1$$

(b) *[5 pts.]* **TRUE or FALSE?** "The complex exponential function is a periodic function with period 2π."

$\boxed{\text{FALSE}}$

The period of e^z is $2\pi i$.

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z$$

Periodic means $f(z+p) = f(z)$ for all z

(c) *[5 pts.]* **TRUE or FALSE?** "$\text{Log}\left(\frac{1}{z}\right) = -\text{Log}(z)$ for all non-zero $z \in \mathbb{C}$."

$\boxed{\text{FALSE}}$

$z = -1 \leftarrow$ counter-example

$$\text{Log}\left(\frac{1}{-1}\right) = \text{Log}(-1) \neq -\text{Log}(-1)$$

$$\text{Log}(-1) = \ln|-1| + i\,\text{Arg}(-1)$$

$$= 0 + i\pi$$

(d) *[5 pts.]* **TRUE or FALSE?** "If $\oint_C f(z)\,dz = 0$ for every contour C lying in a simply connected domain D then $f(z)$ is analytic everywhere in D."

$\boxed{\text{FALSE}}$

This statement RESEMBLES Morera's Theorem, but it lacks the condition that $f(z)$ is continuous everywhere in D.

Cauchy-Goursat Theorem is the converse of this statement [f analytic in $D \Rightarrow \oint_C f(z)dz = 0$ for all C]

(e) *[5 pts.]* **TRUE or FALSE?** "Every entire (i.e. analytic everywhere) function $f(z)$ is the derivative of another entire function."

$\boxed{\text{TRUE}}$

Cool property of analytic functions is that they have an infinite number of derivatives

The generalised CIF tells us that $f^{(n)}(z_0)$ exists and FTCI tells us that an antiderivative exists, so every entire function is the derivative of another.

All the entire functions we know, e^z, $\sin z$, $\cos z$, $\sum_{n=0}^{N} c_n z^n$ have antiderivatives that are also entire.

(f) *[5 pts.]* **TRUE or FALSE?** "The function $\dfrac{\sin(z^4)}{z^4}$ has a removable singularity at $z = 0$."

$\boxed{\text{TRUE}}$

$$\sin z \approx z - \frac{z^3}{3!} + \cdots$$

$$\sin z^4 \approx z^4 - \frac{z^{12}}{3!} + \cdots$$

$$\frac{\sin z^4}{z^4} \approx 1 - \frac{z^8}{3!}$$

$$\lim_{z \to 0} \frac{\sin z^4}{z^4} = 1$$

So this is a removable singularity at 0.

2. *[20 pts. total]* **Laurent Series.** ANALYTICAL, VISUAL, COMPUTATIONAL. Consider the following given Laurent Series about $z = 0$ valid for $|z| > 0$

$$A(z) = \frac{1}{z^5} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z} - \frac{1}{7!}z + \cdots \quad = \quad \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k-5}}{(2k+1)!}$$

$$B(z) = \frac{1}{z^2} - \frac{1}{3!}\frac{1}{z^6} + \frac{1}{5!}\frac{1}{z^{10}} - \frac{1}{7!}\frac{1}{z^{14}} + \cdots \quad = \quad \sum_{k=0}^{\infty}(-1)^k \frac{z^{-4k-2}}{(2k+1)!}$$

$$C(z) = -1 + \frac{1}{3!}z^2 - \frac{1}{5!}z^4 + \frac{1}{7!}z^6 + \cdots \quad = \quad \sum_{k=0}^{\infty}(-1)^{k+1} \frac{z^{2k}}{(2k+1)!}$$

Given the information that if $A(z)$, $B(z)$ and $C(z)$ have singularities, they only occur at $z = 0$ answer the following questions.

(a) *7 points]* Classify the singularities of $A(z)$, $B(z)$ and $C(z)$ at $z = 0$ and give reasons for your classifications.

The degree of the "first term" on the left

$A(z)$ has a pole of order 5 ($\frac{1}{z^5}$ term)

$B(z)$ has a pole of order ∞ (essential singularity) (~~term~~)

$C(z)$ has a removable singularity at 0 $\lim_{z \to 0} C(z) = -1$

(b) *[7 points]* State the values of the residues of $A(z)$, $B(z)$ and $C(z)$ at $z = 0$ and give an explanation for your answers.

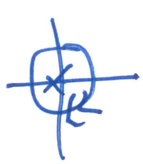
The residue is the coefficient of the $\frac{1}{z}$ term.

$\text{Res}(A,0) = \frac{1}{5!}$ $\qquad$ $\text{Res}(C,0) = 0$

$\text{Res}(B,0) = 0$

(c) *[6 points]* Evaluate $\oint_{|z|=1} A(z) + B(z) + C(z)\, dz$ where the contour is traversed **twice** in a **clockwise** direction

By Cauchy Residue Theorem



$$\oint_{|z|=1} A+B+C\,dz = (-2)\cdot 2\pi i \cdot \left[\overset{=\frac{1}{5!}}{\text{Res}(A,0)} + \overset{=0}{\text{Res}(B,0)} + \overset{=0}{\text{Res}(C,0)}\right]$$

$$= -\frac{4\pi i}{120} = -\frac{\pi i}{30}$$

3. *[25 points. total]* **Application of Contour Integration To Real Integrals, Residues.**
ANALYTIC, COMPUTATIONAL
Considering that $c$and k are positive real numbers, choose one of the integrals below to evaluate.

$$\mathcal{I} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+c^2)^3} \qquad \text{OR} \qquad \mathcal{J} = \int_0^{2\pi} \frac{d\theta}{k-\cos\theta} \quad \text{where } k>1$$

$$f(z) = \frac{1}{(z^2+c^2)^3} = \frac{P(z)}{Q(z)}$$

Clearly $\deg Q = 6$, $\deg P = 1$ $\quad 6-1=5>2$

poles are at $z = \pm ci$ are of order 3

By CRT

$$I = 2\pi i \operatorname{Res}\left(f(z), ci\right)$$

$$\operatorname{Res}\left(\frac{1}{(z+ci)^3}\cdot\frac{1}{(z-ci)^3}\right)$$

$$= \frac{1}{2!}\lim_{z\to ci}\frac{d^2}{dz^2}\frac{1}{(z+ci)^3}$$

$$= \lim_{z\to ci}\frac{1}{2}\frac{d}{dz}\frac{-3}{(z+ci)^4}$$

$$= \lim_{z\to ci}\frac{1}{2}\frac{-3\cdot -4}{(z+ci)^5}$$

$$= \frac{1}{2}\cdot\frac{12}{(2ci)^5}$$

$$= \frac{6}{32c^5 i}$$

$$I = \frac{12\pi}{32c^5} = \frac{3\pi}{8c^5}$$

$z = e^{i\theta} \quad |z| = 1 \quad dz = iz\,d\theta$

$$J = \oint_{|z|=1} \frac{1}{k - \frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}$$

$$= \frac{1}{i}\oint \frac{2}{2kz - z^2 - 1}\,dz$$

$$= -\frac{1}{i}\oint \frac{2}{z^2 - 2kz + 1}\,dz$$

$$z^2 - 2kz + 1 = 0$$

$$z = \frac{2k \pm \sqrt{(2k)^2 - 4(1)(1)}}{2}$$

$$= \frac{2k \pm \sqrt{4k^2-4}}{2}$$

$$= k \pm \sqrt{k^2-1} \qquad k>1$$

Only root inside $|z|=1$ is $k - \sqrt{k^2-1}$

$$J = 2\pi i\left(-\frac{1}{i}\right)\operatorname{Res}\left(\frac{2}{z^2-2kz+1}, k-\sqrt{k^2-1}\right)$$

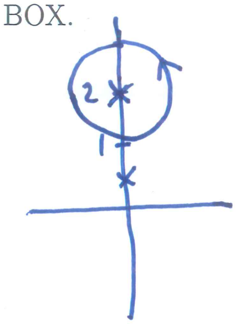
$$\text{Residue} = \frac{2}{2z-2k} = \left.\frac{1}{z-k}\right|_{z=k-\sqrt{k^2-1}}$$

$$= \frac{1}{-\sqrt{k^2-1}}$$

$$J = \frac{2\pi}{\sqrt{k^2-1}}$$

4. *[25 points. total]* **Cauchy's Integral Theorems, Residues.**
ANALYTIC, COMPUTATIONAL, VISUAL, VERBAL.
Evaluate the following integrals. All contours are closed and traversed once in the counter-clockwise direction. **STATE what theorem and/or formula you are using and SKETCH the location of the contour and any poles for each problem you evaluate below.** WRITE THE VALUE OF EACH INTEGRAL IN THE BOX.

(a) *[6 points]* $\oint_{|z-2i|=1} \frac{z}{(2z-i)(z-2i)}\, dz = \boxed{4\pi i/3}$

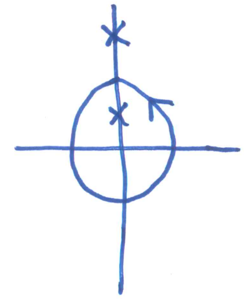


Simple pole at $z = 2i$

$\text{Res}\left(\frac{z}{(2z-i)(z-2i)}, 2i\right) = \frac{2i}{4i-i} = \frac{2}{3}$

By Caucy Residue Theorem

(b) *[6 points]* $\oint_{|z|=1} \frac{z}{(2z-i)(z-2i)}\, dz = \boxed{-\pi i/3}$

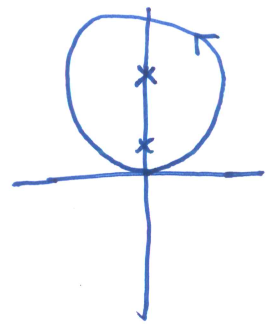


Simple pole at $z = \frac{i}{2}$

$\text{Res}\left(\frac{z}{2(z-\frac{i}{2})(z-2i)}, \frac{i}{2}\right) = \frac{i/2}{2(\frac{i}{2}-2i)} = \frac{i/2}{2\cdot -\frac{3i}{2}} = -\frac{1}{6}$

By CRT

(c) *[6 points]* $\oint_{|z-2i|=2} \frac{z}{(2z-i)(z-2i)}\, dz = \boxed{\pi i}$

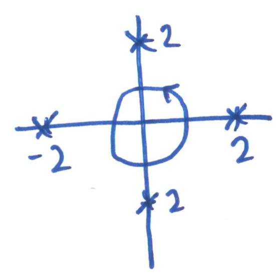


Both poles inside

Answer is sum of previous residues

$2\pi i\left(\frac{2}{3} - \frac{1}{6}\right) = 2\pi i\left(\frac{3}{6}\right) = \pi i$

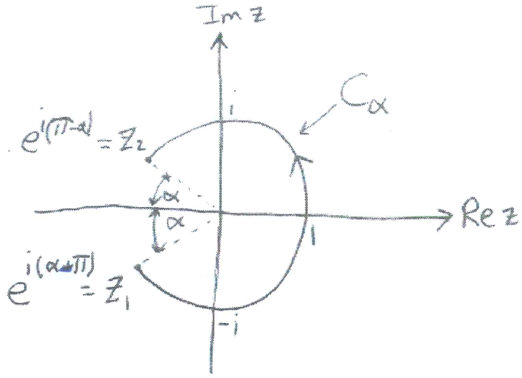
(d) *[7 points]* $\oint_{|z|=1} \frac{1}{z^4 - 16}\, dz = \boxed{0}$



By Cauchy-Goursat Theorem

integrand is analytic inside and on the contour.

EXTRA CREDIT *[5 pts.]* Consider the contour C_α shown in the figure below:



$Z_2 = e^{i(\pi-\alpha)}$

$Z_1 = Z_2^{*} = e^{-i(\pi-\alpha)} = e^{i(\alpha-\pi)}$

C_α is almost the entire unit circle $|z| = 1$ except for a sector of size 2α radians symmetric about the negative horizontal Re(z)-axis. When $f(z) = \mathrm{Log}(z)$, the principal branch of the complex log function which is analytic on its domain,

(a) Evaluate $I_\alpha(f) = \int_{C_\alpha} f(z)\, dz$, i.e. $\int_{C_\alpha} \mathrm{Log}(z)\, dz$.

(b) Evaluate $I(f) = \oint_{|z|=1} f(z)\, dz$, i.e. $\oint_{|z|=1} \mathrm{Log}(z)\, dz$.

(c) Show that $\lim_{\alpha\to 0} \int_{C_\alpha} \mathrm{Log}(z)\, dz = \oint_{|z|=1} \mathrm{Log}(z)\, dz$.

(a) Log(z) is analytic in an open set containing C_α so you can use Fundamental Theorem! $F'(z) = \mathrm{Log}\, z$ so $F(z) = z\,\mathrm{Log}\, z - z$

$$\begin{aligned} I_\alpha(f) &= F(z_2) - F(z_1) \\ &= z_2 \mathrm{Log}\, z_2 - z_2 - [z_1 \mathrm{Log}\, z_1 - z_1] \\ &= z_2 \mathrm{Log}\, z_2 - z_1 \mathrm{Log}\, z_1 + z_1 - z_2 \\ &= e^{i(\pi-\alpha)} \mathrm{Log}(e^{(\pi-\alpha)i}) - e^{i(\alpha-\pi)} \mathrm{Log}(e^{i(\alpha-\pi)}) + e^{i(\alpha-\pi)} - e^{i(\pi-\alpha)} \\ &= -e^{-\alpha i}(\pi-\alpha)i + e^{\alpha i}(\alpha-\pi)i - e^{\alpha i} + e^{-\alpha i} \\ &= e^{\alpha i}[-1+(\alpha-\pi)i] + e^{-\alpha i}[1-(\pi-\alpha)i] \end{aligned}$$

(c) $\lim_{\alpha\to 0} I_\alpha(f) = \lim_{\alpha\to 0} e^{\alpha i}[-1+(\alpha-\pi)i] + e^{-\alpha i}[1-(\pi-\alpha)i] = 1(-1-\pi i) + 1(1-\pi i)$

$= -1-\pi i + 1 - \pi i$

$= -2\pi i$

(b) $I(f) = \oint_{|z|=1} \mathrm{Log}\, z\, dz = \int_{-\pi}^{\pi} \mathrm{Log}(e^{it})\, i e^{it}\, dt = \int_{-\pi}^{\pi} -t e^{it}\, dt$ $\quad u = t \quad dv = e^{it}$, $du = dt \quad v = \frac{e^{it}}{i}$

$z(t) = e^{it} \quad z' = ie^{it}$

$I(f) = -\left[\frac{te^{it}}{i}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{it}}{i} dt \right] = -\left[\frac{\pi e^{i\pi}}{i} - (-\pi)\frac{e^{-\pi i}}{i} - \frac{e^{it}}{i^2}\Big|_{-\pi}^{\pi} \right]$

$= -[\pi i + \pi i - 0] = -2\pi i$