Complex Analysis

Math 312 Fall 2001 | M 5-6:25, R 1:30-2:55

 | Fowler 112, Fowler 201

TEST 1: Friday October 19, 2001

Directions: Read *all* 4 problems first before answering any. **You may choose to answer question 3 or question 4.** You may have access to any notes or the textbook. This is a one hour test. You have 90 minutes to complete it.

No.	Score	Maximum
1		40
2		30
3 or 4		30
Total		100

1. *[40 pts. total]* **Mapping.** We want to find the general form of the function $M(z) = Az + B$ which maps one circle, the set of points **P**: $|z - z_0| = r$, to another circle located somewhere else, the set of points **Q**: $|w - w_0| = \rho$ in the complex plane.

(a) *[5 pts]* Find a mapping of the form $f_1(z) = \alpha z$ which maps **P** so that it has the same radius as **Q**.

(b) *[5 pts]* Find a mapping of the form $f_2(z) = z + \beta$ which maps **P** so that its center is at the same location as **Q**.

(c) *[10 pts]* Will $f_2(f_1(z)) = F(z)$ be the mapping which maps **P** to **Q**? In other words, what is the image of **P** under $F(z)$?

(d) *[10 pts]* Find an example of a mapping $M(z) = Az + B$ where A and B depend on the parameters r, ρ, z_0 and w_0 which maps $\mathbf{P}$ to $\mathbf{Q}$.

(e) *[10 pts]* Use your answers above to find the function $M(z)$ which maps $|z - 2 - i| = 1$ to $|w + 2 + 3i| = 2$.

2. *[30 pts.]* **Arithmetic of Complex Numbers**. **(a)** *[10 pts]* What condition on a and b must be met for $(a + bi)^2 = ci$ where a, b and c are all real numbers? Where in the complex plane would (a, b) have to be for c to be negative?

(b) *[10 pts]* Use your answer in part **(a)** to help you evaluate $\sqrt{8i}$.

(c) *[10 pts]* Use your answers above to find all the solutions of $z^2 + 2iz + 8i - 1 = 0$.

3. *[30 pts. total]* **Cauchy-Riemann Equations, Harmonic Functions**. Consider the function $f = u(x, y) + iv(x, y) = x^2 + y^2 + 2xyi$

(a) *[10 pts]* Show that the set of points for which $f'(z)$ exists all lie on the x-axis.

(b) *[10 pts]* Using your information from **(a)**, on what set of points is $f(z)$ analytic? Explain your answer.

(c) *[10 pts]* Show that $v(x, y)$ is harmonic. Is the given $u(x, y)$ its harmonic conjugate? If not, find the harmonic conjugate of $v(x, y)$

4. *[30 pts. total]* **Analyticity, Differentiability**.

The **Jacobian** of a mapping $u = u(x, y), v = v(x, y)$ from the xy-plane to the uv-plane is defined to be the determinant

$$J(x_0, y_0) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

where the partial derivatives u_x, u_y, v_x, v_y are all evaluated at (x_0, y_0).

(a) *[10 pts]* If $f = u + iv$ is analytic on a neighborhood containing $z_0 = x_0 + iy_0$ **show** that $J(x_0, y_0) = |f'(z_0)|^2$.

(b) *[20 pts]* For the function $f(z) = Az + B$ find $J(0, 0)$ two different ways (i.e. from the definition and from the result given in part **(a)**.