# Test 1: Complex Analysis

Math 312 Spring 2016 February 29, 2016



# Name:

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**Directions**:

Read *all* problems first before answering any of them. This tests consists of three (3) problems (and a BONUS problem) on eight (8) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, limited-notes*, closed book, test. **No calculators or electronic devices may be used.**

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

***You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam when you hand it in.**

**FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!**

**Pledge:** I, ________________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

| No. | Score | Maximum |
|---|---|---|
| 1 | | 28 |
| 2 | | 32 |
| 3 | | 40 |
| EXTRA CREDIT | | 5 |
| **Total** | | **100** |

**1.** *[28 pts. total]* **Analyticity, Cauchy-Riemann Equations, Cauchy-Schwarz Inequality, Complex Functions of a Complex Variable**. ANALYTIC, COMPUTATIONAL, VERBAL. Determine whether the following statements are **TRUE** or **FALSE** and place your answer in the box. To receive FULL credit, you must also give a very brief (and correct) explanation in support of your TRUE/FALSE choice! The explanation for your answer is worth SIX (6) POINTS while your correct TRUE or FALSE answer is worth ONE (1) point.

**(a)** *[7 points]*. **TRUE or FALSE**: "The function $f(z) = x^2 + y^2 + 2xyi$ is not analytic at any point in the Complex plane."

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**(b)** *[7 points]*. **TRUE or FALSE**: "The function $\text{Arg}(z)$ maps the entire complex plane to a subset of the real numbers."

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**(c)** *[7 points]*. **TRUE or FALSE**: "If $|z| \leq 2$, then $|z^2 - 2iz - 1| \leq 9$."

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**(d)** *[7 points]*. **TRUE or FALSE**: "The set of points $\mathcal{A} = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$ is both open and closed."

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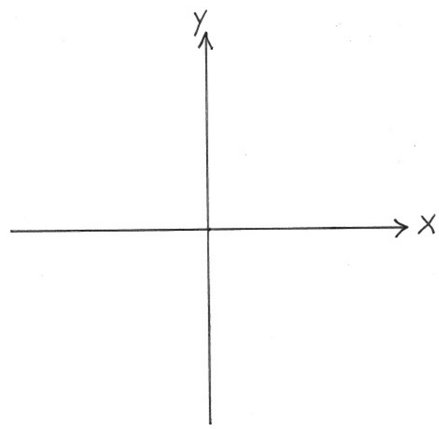
**2.** *[32 pts. total]* **Operations on Complex Numbers, Argand Plane, Visualization.** VISUAL, ANALYTIC, COMPUTATIONAL, VERBAL. Consider the specific complex number $Z = a(1+i)$, written as $(x, y)$ coordinates of a point in the Argand plane as $Z = (a, a)$ where $a$ **is a specific (fixed) positive real number such that** $2 < a < 3$.

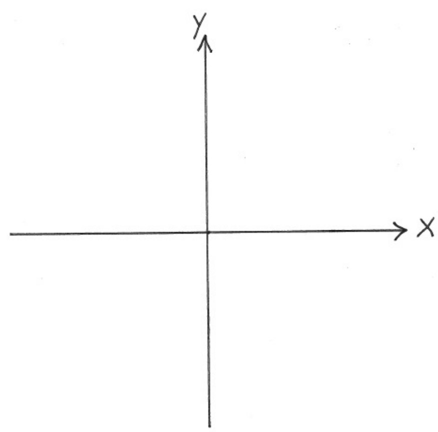
On each of the axes below, sketch the associated complex number $W$ in the complex plane. (You will get 4 points for drawing the correct geometric location of $W$ and $Z$, 2 points for the correct coordinates of $W$ in each case written in the box and 2 points for a short explanation and/or illustrative work which supports your answers).

**ON EACH FIGURE DRAW THE LOCATION OF $W$ AND $Z$.**
**ADD THE UNIT CIRCLE CENTERED AT THE ORIGIN FOR SCALE.**
**WRITE THE COORDINATES FOR $W$ IN THE BOX.**

**(a)** *[8 points]* $W = \overline{Z}$

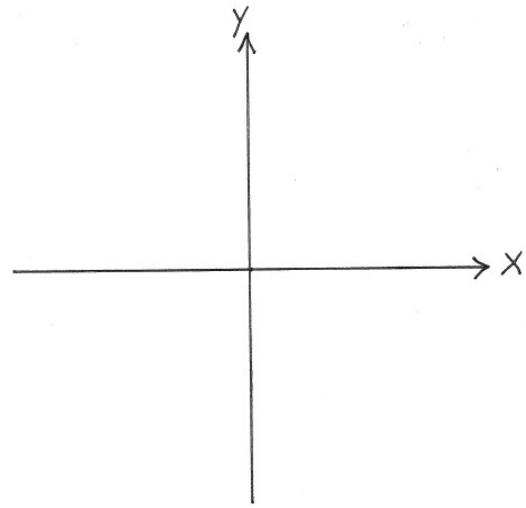



**(b)** *[8 points]* $W = \dfrac{1}{Z}$

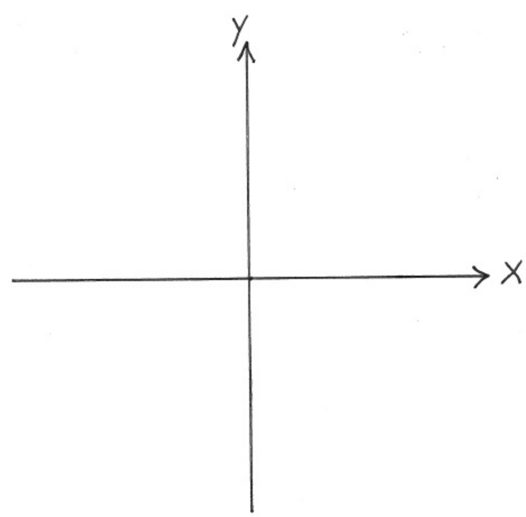


**ON EACH FIGURE DRAW THE LOCATION OF $W$ AND $Z$.**
**ADD THE UNIT CIRCLE CENTERED AT THE ORIGIN FOR SCALE.**
**WRITE THE COORDINATES FOR $W$ IN THE BOX.**

**(c)** *[8 points]* $W = Z^2$




**(d)** *[8 points]* $W^2 = Z$



**3.** *[40 pts. total]* **Mapping, Reciprocal Function, Composition, Point Sets in the Complex Plane**. ANALYTICAL, VISUAL, COMPUTATIONAL. One of the most important class of mappings in Complex Analysis is the class of functions called linear fraction transformations (LFT). $w = B(z) = \dfrac{1-z}{1+z}$ is an example of an LFT. Every LFT can be thought of as a composition of two linear mappings and a reciprocal mapping.

**(a)** *[4 pts]* Use algebra to show that $B(z) = \dfrac{1-z}{1+z} = -1 + 2\dfrac{1}{1+z}$.

**(b)** *[4 pts]* Given $f_3(z) = -1 + 2z$, $f_2(z) = \dfrac{1}{z}$ and $f_1(z) = 1 + z$ use algebra to **show that** $B(z)$ **can be written as the composition of these two linear mappings and the reciprocal mapping**, i.e. $B(z) = (f_3 \circ f_2 \circ f_1)(z)$ or $B(z) = f_3(f_2(f_1(z)))$.

**(c)** *[4 pts]* Show that the the image of the set $\mathcal{P} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ under the mapping $w_1 = f_1(z)$ is the set $\mathcal{Q} = \{w_1 \in \mathbb{C} \ : \ |w_1 - 1| = 1\}$. **Describe the pre-image** $\mathcal{P}$ **and image** $\mathcal{Q}$ **in words** and **categorize in general terms the effect** $f_1$ **has on a set of points** (i.e. as some combination of rotation, scaling, translation).

**(d)** *[8 pts]* Recall that the inversion mapping $w = \dfrac{1}{z}$ maps lines given by $\text{Re}(z) = k$ to circles given by $\left|w - \dfrac{1}{2k}\right| = \left|\dfrac{1}{2k}\right|$ for all $k \neq 0$. Using this information and the idea that the reciprocal function is its own inverse, **what is the image $\mathcal{R}$ of the set $\mathcal{Q} = \{w_1 \in \mathbb{C} : |w_1 - 1| = 1\}$ under the mapping $w_2 = f_2(w_1) = \dfrac{1}{w_1}$? Describe $\mathcal{R}$ in point set notation.**

**(e)** *[4 pts]* Using your answer from **(d)**, show that $\mathcal{S} = \{w \in \mathbb{C} : \text{Re}(w) = 0\}$ is the image of $\mathcal{R}$ under the mapping $w = f_3(w_2) = -1 + 2w_2$. **Describe the pre-image $\mathcal{R}$ and image $\mathcal{S}$ in words** and **categorize in general terms the effect $f_3$ has on a set of points** (i.e. as some combination of rotation, scaling, translation).

**(f)** *[12 pts]* Draw pictures which demonstrate the action of $w = B(z) = \dfrac{1-z}{1+z}$ on the set of points $|z| = 1$ in the space below by showing the action of each constituent mapping (i.e. $f_1$ then $f_2$ then $f_3$ on $|z| = 1$). I expect you to do this by **sketching the curves $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ on four separate, clearly labelled copies of the complex plane** (i.e. $z$, $w_1$,$w_2$ and $w$) in the space below.

**(g)** *[4 pts]* **Indicate on your sketch above where the interior of the unit disk $|z| \leq 1$ gets mapped to under** $B(z)$. (HINT: Pick a point $\mathcal{X}$ inside $|z| \leq 1$ and show where it moves under each constituent mapping $f_1$ then $f_2$ then $f_3$ from the $z$-plane to the $w_1$-plane to the $w_2$-plane to the $w$-plane on your four axes drawn above.)

**BONUS** *[5 pts.]*

This BONUS problem is about deriving the Cauchy-Riemann Equations in polar coordinates. Consider $f(z) = u(x, y) + iv(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ so that $f(z) = u(r, \theta) + iv(r, \theta)$.

**(a)** *[2 BONUS POINTS.]* Use the multivariable chain rule to obtain expressions for $u_r$ in terms of $u_x$ and $u_y$, $v_r$ in terms of $v_x$ and $v_y$, $u_\theta$ in terms of $u_x$ and $u_y$, and $v_\theta$ in terms of $v_x$ and $v_y$.

**(b)** *[3 BONUS POINTS.]* By substituting the equations $u_x = v_y, \quad u_y = -v_x$ into your expressions for $v_r$ and $v_\theta$ and comparing the results with your expressions for $u_r$ and $u_\theta$ you should be able to deduce the Cauchy-Riemann Equations in polar coordinates.