Final Exam

Math 312 Spring 2004 **Complex Analysis**

Thursday, May 6, 2004: 1–4pm Prof. R. Buckmire

Name: BUCKMIRE

Directions: Read *all* problems first before answering any of them. There are EIGHT (8) problems on NINE (9) pages.

This exam is a limited-notes, closed-book, test. You may use a calculator and bring in one 8.5 x 11 inch sheet of paper.

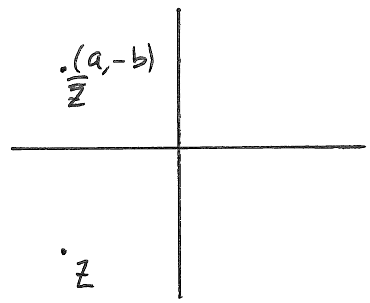
You must include ALL relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answer from your "scratch work.'

No.	Score	Maximum
1.		**25**
2.		**25**
3.		**25**
4.		**25**
5.		**25**
6		**25**
7.		**25**
8.		**25**
TOTAL		**200**

1. *[25 pts.]* **Operations on Complex Numbers, Visualization.** Consider the specific complex number $Z = a + bi$, sometimes written, $Z = (a, b)$. Graphically, Z can be depicted in the third quadrant of complex plane, outside the unit circle (see Figures). On each of the axes below, sketch the associated complex number W in the plane. (You should write a one-sentence explanation for how you know where to draw each of the associated numbers and give the coordinates of W in each case.)

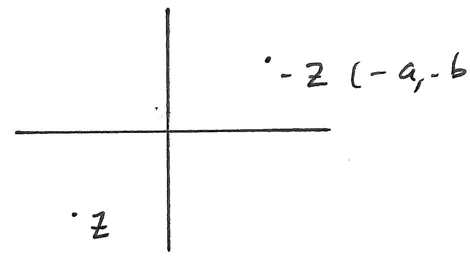
(a) *[5 points]* $W = \overline{Z}$ $= a - bi$

W is reflected across x-axis to a position equidistant from origin



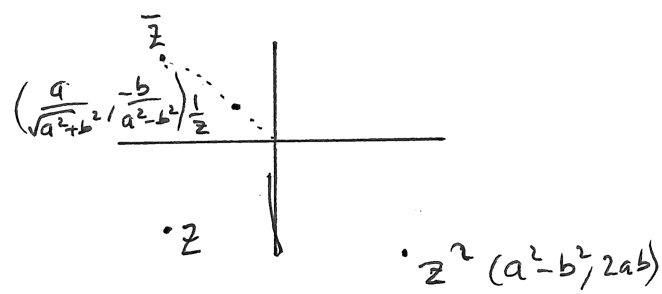
(b) *[5 points]* $W = -Z$ $= -a-bi$

W is rotated 180° from 3rd quadrant to 1st quadrant, equidistant from origin as Z.



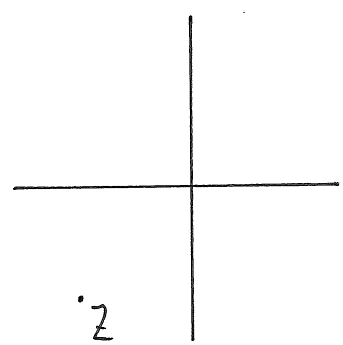
(c) *[5 points]* $W = \dfrac{1}{Z}$ $= \dfrac{\overline{Z}}{|Z|^2} = \dfrac{a-bi}{\sqrt{a^2+b^2}}$

If $|Z|>1$ then W is on same ray as $\overline{Z}$, but closer to origin



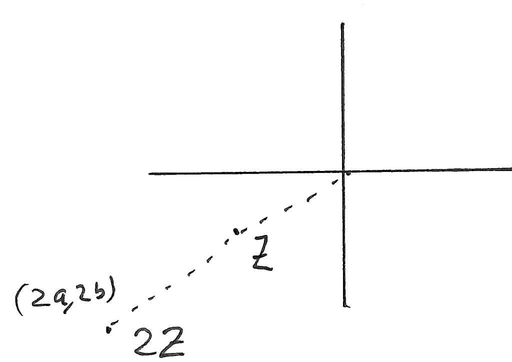
(d) *[5 points]* $W = Z^2$ $= a^2 - b^2 + 2abi$

Moves from 3rd Quadrant to 1st Quadrant. If $|Z|>1$ then $|Z^2| > |Z| > 1$ further away from origin



(e) *[5 points]* $W = 2Z$ $= 2a+2bi$

W Doubles the distance from the origin, Keeps the same argument as Z



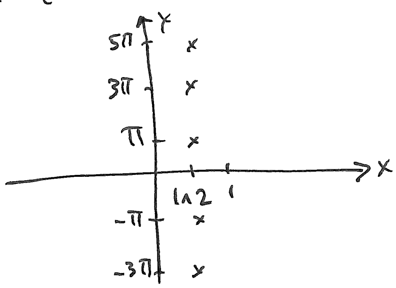
2. *[25 pts.]* **Complex Arithmetic, Elementary Functions.** Find the complex numbers which satisfy the equations below. **Be confident of your solutions, because you will use your answers in subsequent problems.** In each case, sketch the points in the complex plane which solve the equations below.

(a) *[8 points]* Find all solutions of the equation $e^z + 2 = 0$.

$$e^z = -2$$

$$z = \log(-2)$$

$$z = \ln 2 + \pi i + 2k\pi i, \quad k \in \mathbb{Z}$$



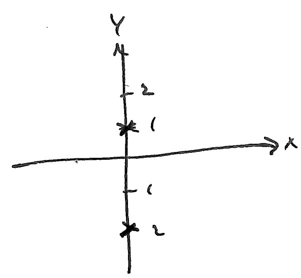
(b) *[8 points]* Find all solutions of the equation $z^2 + iz + 2 = 0$

$$z = \frac{-i \pm \sqrt{i^2 - 4(2)}}{2}$$

$$= \frac{-i \pm \sqrt{-1-8}}{2}$$

$$= \frac{-i \pm 3i}{2}$$

$$= \frac{-4i}{2}, \frac{2i}{2} = -2i, i$$



(c) *[9 points]* Find all solutions of the equation $z^6 + 1 = 0$ (Write your answer in exponential form.)

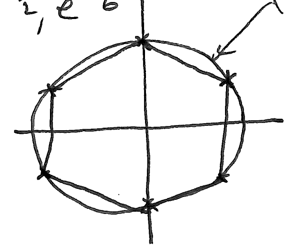
$$z^6 = -1 = e^{\pi i} = e^{\pi i + 2k\pi i}, \quad k \in \mathbb{Z}$$

$$z = e^{\frac{\pi i + 2k\pi i}{6}}$$

$$= \cancel{e^{\frac{\pi i}{6} + k}} \; e^{\frac{\pi i}{6}(2k+1)}, \quad k = 0, 1, 2, 3, 4, 5$$

$$z = e^{\frac{\pi i}{6}}, e^{\frac{\pi i}{2}}, e^{\frac{5\pi i}{6}}, e^{-\frac{\pi i}{6}}, e^{-\frac{\pi i}{2}}, e^{-\frac{5\pi i}{6}}$$

$|z| = 1$



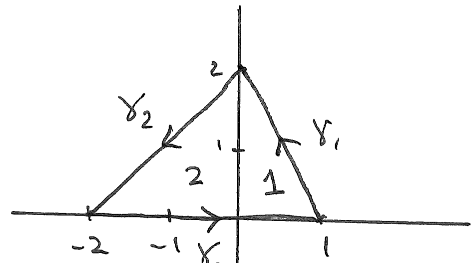
3. *[25 pts.]* **Contour Integration, Parametrization.**

(a) *[15 points]* Using Contour Integration, evaluate $\oint_\Gamma \bar{z}\, dz$ where Γ is the contour consisting of the three straight line segments connecting the points (1,0), (0,2), (-2,0), traversed in a counter-clockwise direction.

$\gamma_1 : 1 \to 2i \qquad z_1(t) = (2i-1)t + 1, \; t: 0 \to 1$

$\gamma_2 : 2i \to -2 \qquad z_2(t) = (-2-2i)t + 2i, \; t: 0 \to 1$

$\gamma_3 : -2 \to 1 \qquad z_3(t) = t, \; t: -2 \to 1$



$$\oint_\Gamma \bar{z}\,dz = \int_{\gamma_1} \bar{z}\,dz + \int_{\gamma_2} \bar{z}\,dz + \int_{\gamma_3} \bar{z}\,dz$$

$$= 2i + \frac{3}{2} + 4i - \frac{3}{2} = \boxed{6i} = 2i\,(\text{area of triangle}) = 2i \cdot 3$$

$$\int_{\gamma_1} \bar{z}\,dz = \int_0^1 [(-2i-1)t+1](2i-1)\,dt = \int_0^1 2i - 1 + t\left((-1)^2 - (2i)^2\right) dt$$

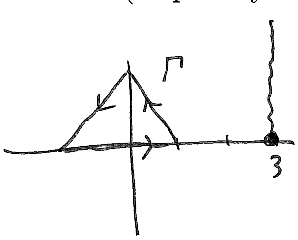
$$= (2i-1)t\Big|_0^1 + \int_0^1 t\,5\,dt = 2i - 1 + 5 \cdot \frac{1}{2} = \boxed{2i + \frac{3}{2}}$$

$$\int_{\gamma_2} \bar{z}\,dz = \int_0^1 [(-2+2i)t - 2i](-2-2i)\,dt = \int_0^1 4i - 4\,dt + \int_0^1 \left((-2)^2 - (2i)^2\right) t\,dt$$

$$= 4i - 4 + \int_0^1 8t\,dt = 4i - 4 + 4t^2\Big|_0^1 = 4i - 4 + 4 = \boxed{4i}$$

$$\int_{\gamma_3} \bar{z}\,dt = \int_{-2}^1 t \cdot 1 \cdot dt = \frac{t^2}{2}\Big|_{-2}^1 = \frac{1}{2} - \frac{(-2)^2}{2} = \frac{1}{2} - 2 = \boxed{-\frac{3}{2}}$$

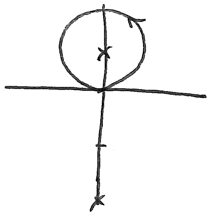
(b) *[10 points]* Evaluate $\oint_\Gamma \log_{\pi/2}(z-3)\, dz$ where Γ is the same contour given in part (a). (Explain your answer!)



No poles or singularities of $\log_{\pi/2}(z-3)$ lie inside Γ so by Cauchy-Goursat, $\oint_\Gamma \log_{\pi/2}(z-3)\,dz = 0$

4. *[25 pts. total]* **Cauchy's Integral Theorems**. Evaluate the following integrals. All contours are closed and traversed once in the counter-clockwise direction.

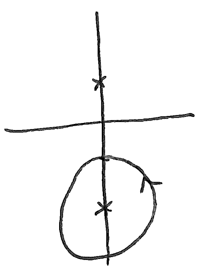
(a) *[6 points]* Evaluate $\oint_{|z-i|=1} \frac{1}{z^2+iz+2}\, dz$ $= \oint \frac{1}{(z-i)(z+2i)}\, dz$



CIF $= \oint \frac{\frac{1}{z+2i}}{z-i}\, dz$

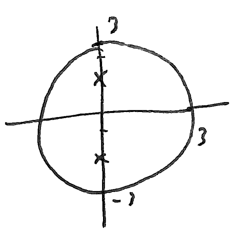
$= 2\pi i \frac{1}{i+2i} = \frac{2\pi i}{3i} = \boxed{\frac{2\pi}{3}}$

(b) *[6 points]* Evaluate $\oint_{|z+2i|=1} \frac{1}{z^2+iz+2}\, dz$ $= \int \frac{\frac{1}{z-i}}{z+2i}\, dz$



CIF $= 2\pi i \frac{1}{-2i-i} = \frac{2\pi i}{-3i} = \boxed{-\frac{2\pi}{3}}$

(c) *[6 points]* Evaluate $\oint_{|z|=3} \frac{1}{z^2+iz+2}\, dz$ $= 2\pi i \left(\frac{1}{3i} - \frac{1}{3i} \right)$ CRT 1



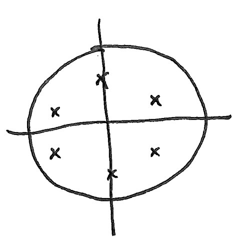
$= \boxed{0}$

(d) *[7 points]* Evaluate $\oint_{|z|=2} \frac{1}{z^6+1}\, dz$ $= 2\pi i \operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0 \right) = 2\pi i \cdot 0$

$= \boxed{0}$

CRT 2

$f(z) = \frac{1}{z^6+1}$



$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{1}{\left(\frac{1}{z}\right)^6 + 1} = \frac{1}{z^2} \frac{z^6}{1+z^6} = \frac{z^4}{1+z^6}$

no pole at $z=0$

5. *[25 pts. total]* **Harmonic Conjugates and Analytic Functions.**
Given that $u(x,y) = x^2 - y^2 + 2y + 1$ and $f(i) = 2 + 2i$ we want to find the analytic function $f(z) = u(x,y) + iv(x,y)$ and compute $f'(i)$.

(a) *[5 pts]* Show that $u(x,y)$ is harmonic.

$u_x = 2x \qquad u_{xx} = 2 \qquad\qquad u_{xx} + u_{yy} = 2 - 2 = 0$ ✓

$u_y = -2y + 2 \qquad u_{yy} = -2$

(b) *[15 pts]* Use the CREs to find $v(x,y)$, the harmonic conjugate of $u(x,y)$

$u_x = v_y = 2x \Rightarrow v(x,y) = 2xy + f(x)$

$u_y = -v_x \Rightarrow -2y + 2 = -2y + f'(x) \Rightarrow -f'(x) = 2$

$f'(x) = -2$

$f(x) = -2x + C$

$v(x,y) = 2xy - 2x + C$

$v(x,y) = 2xy - 2x + 2$

(c) *[10pts]* Use the information that $f(i) = 2 + 2i$ to write $f(z)$ explicitly (i.e. no unknown constants and in z-variables). Confirm that $v(x,y)$ is harmonic.

$f(z) = u(x,y) + iv(x,y)$

$f(i) = u(0,1) + iv(0,1) = 2 + 2i$

$= C + 2i \qquad = 2 + 2i \Rightarrow C = 2$

$f(z) = x^2 - y^2 + 2y + 1 + i(2xy - 2x + 2)$

$= x^2 - y^2 + 2xyi - 2i(x + iy) + 1 + 2i$

$= z^2 - 2iz + 1 + 2i$

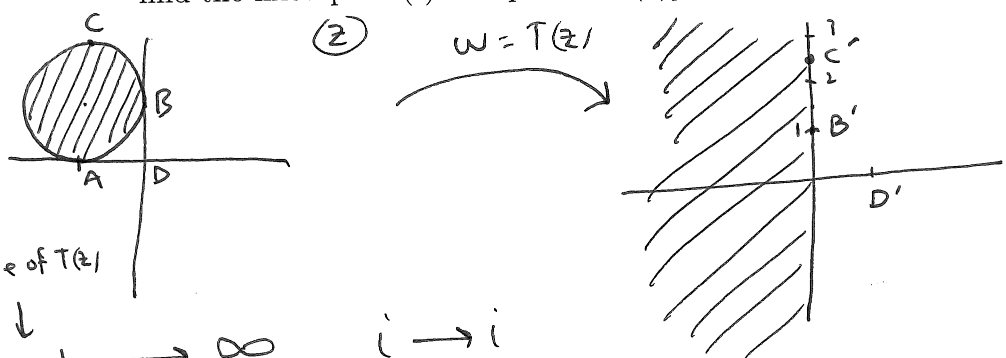
(d) *[5 pts]* Compute the value of $f'(i)$

$f'(z) = 2z - 2i$

$f'(i) = 0$

6. *[25 pts. total]* **LFTs, Mapping.** Consider the bilinear transformation $w = T(z) = \dfrac{(1+2i)z+1}{z+1}$. We want to find the function which maps $|z+1-i| \leq 1$ to the upper half-plane.

(a) *[10 pts]* Show that the image of the disk $|z+1-i)| \leq 1$ under $w = T(z)$ is Re $(z) \leq 0$. Show as much work so that it is clear how you have computed your answer. [HINT: find the fixed point(s) and pole of $T(z)$]



$-1+2i \xrightarrow{T} \frac{5i}{2}$

$-1 \xrightarrow{T} \infty$

$i \xrightarrow{T} i$

$0 \xrightarrow{T} 1$

pole of T(z)

$-1 \longrightarrow \infty$ $\quad$ $i \longrightarrow i$

fixed pt

$$\frac{(1+2i)z+1}{z+1} = z$$

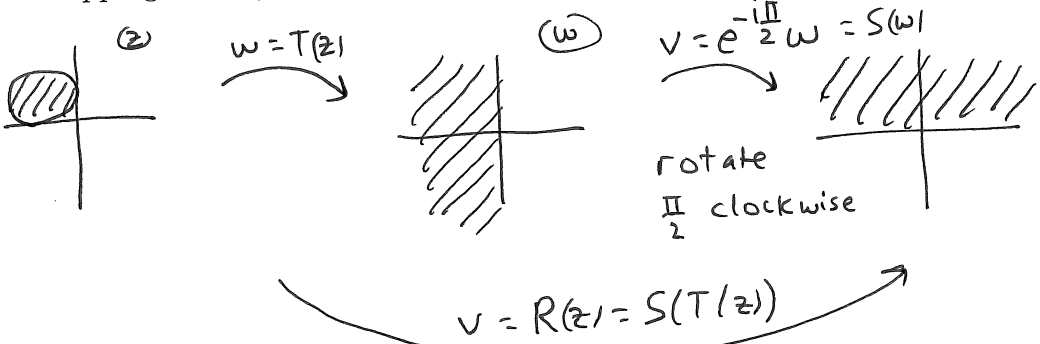
$$z + 2iz + 1 = z^2 + z$$

$$0 = z^2 - 2iz - 1$$

$$0 = (z-i)^2 \Rightarrow z = i \text{ is fixed pt}$$

$$T(-1+2i) = \frac{(1+2i)(-1+2i)+1}{-1+2i+1} = \frac{-1+2i-4+2i}{2i} = -\frac{5}{2i} = \frac{5i}{2}$$

(b) *[15 pts]* Use your information from above to write down a bi-linear mapping $v = R(z)$ which takes the interior of the disk $|z+1-i| \leq 1$ to the upper half-plane. [HINT: map the image of the disk $|z+1-i| \leq 1$ under $w = T(z)$ to the upper half-plane using a mapping $v = S(w)$ and then combine T and S]



$-1 \xrightarrow{R} \infty$

$0 \xrightarrow{R} -i$

$i \xrightarrow{R} 1$

$-1+2i \xrightarrow{R} 2$

$$v = S(T(z)) = (S \circ T)(z) = e^{-\frac{i\pi}{2}} T(z) = -i\,T(z)$$

$$= -i\left[\frac{(1+2i)z+1}{z+1}\right] = \frac{(2-i)z-i}{z+1} = R(z)$$

pole of R is -1

$-1 \longrightarrow \infty$

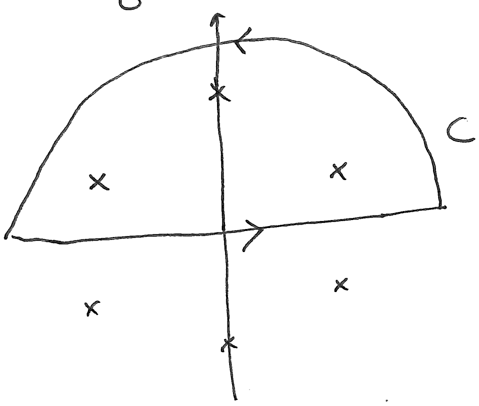
$0 \longrightarrow -i$

$$R(i) = \frac{(2-i)i-i}{1+i} = \frac{1+i}{1+i} = 1$$

$$R(-1+2i) = \frac{(2-i)(-1+2i)-i}{2i} = \frac{i-2+4i+2-i}{2i} = 2$$

7. *[25 pts. total]* **Application of Residues**. Show that $\displaystyle\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$ by evaluating a contour integral.

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\oint_C \frac{dz}{z^6+1} = \pi i\left[\mathrm{Res}\left(e^{\frac{i\pi}{6}}\right) + \mathrm{Res}\left(e^{\frac{5i\pi}{6}}\right) + \mathrm{Res}(i)\right]$$



Only 3 of the 6 poles are inside the contour. Can't use CRT² because there are poles outside

$$\mathrm{Res}\left(e^{\frac{i\pi}{6}}\right) = \lim_{z\to e^{\frac{i\pi}{6}}} \frac{z-e^{\frac{i\pi}{6}}}{z^6+1} \overset{L'H}{=} \lim_{z\to e^{i\pi/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6e^{\frac{5\pi i}{6}}}$$

$$\mathrm{Res}\left(e^{\frac{5\pi i}{6}}\right) = \lim_{z\to e^{\frac{5\pi i}{6}}} \frac{z-e^{\frac{5\pi i}{6}}}{z^6+1} \overset{L'H}{=} \frac{1}{6z^5}$$

$$= \frac{1}{6e^{\frac{25\pi i}{6}}} = \frac{1}{6e^{\frac{\pi i}{6}}}$$

$$\mathrm{Res}(i) = \lim_{z\to i} \frac{z-i}{z^6+1} \overset{L'H}{=} \lim_{z\to i} \frac{1}{6z^5} = \frac{1}{6i^5}$$

$$= \frac{1}{6i}$$

$$\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{1}{6}\left(e^{-\frac{5\pi i}{6}} + e^{-\frac{\pi i}{6}} - i\right)\pi i$$

$$= \frac{\pi i}{6}\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right) - i\right]$$

$\cos(\pi-\alpha) = -\cos\alpha$

$\sin(\pi-\alpha) = \sin\alpha$

$$= \frac{\pi i}{6}\left[-\frac{i}{2} - \frac{i}{2} - i\right]$$

$$= \frac{\pi i}{6}(-2i) = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$$

8. *[25 points.]* **TRUE/FALSE.** Consider the following statements and fillin the box with either TRUE or FALSE for each statement below. To be true, the statement must ALWAYS be true. If you think the statement is FALSE, give an example of a which supports this view. If you think the statement is TRUE, you should provide work which supports this view. You will receive *1 point* for your choice of TRUE/FALSE and *4 points* for your explanation or counterexample.

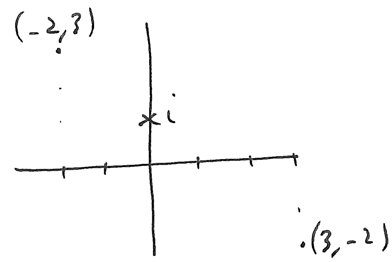
(a) In the Complex Plane, $-2+3i$ is closer to i than $3-2i$.

TRUE

$|(-2+3i)-i| = |-2+2i| = 2|-1+i| = 2\sqrt{2}$

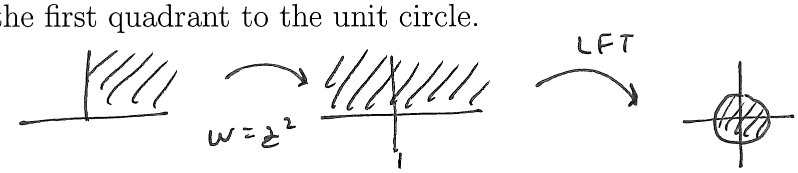
$|(3-2i)-i| = |3-3i| = 3|1-i| = 3\sqrt{2}$

$3\sqrt{2} > 2\sqrt{2}$



(b) There is no way to map the first quadrant to the unit circle.

FALSE



(c) $\log_{2\pi}(i^2) = 2\log_{2\pi}(i)$

FALSE

$\log_{2\pi}(-1) = \ln 1 + i\,\mathrm{Arg}_{2\pi}(-1) = i3\pi$

$2\log_{2\pi}(i) = 2[\ln 1 + i\,\mathrm{Arg}_{2\pi}(i)] = 2i\,\mathrm{Arg}_{2\pi}(i)$

$= 2i\left(2\pi + \frac{\pi}{2}\right) = 5\pi i \neq 3\pi i$

(d) The function $\dfrac{\sin(z)}{z^6}$ has a pole of order 5 at $z = 0$.

TRUE

$\sin z \approx z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots$

$\frac{\sin z}{z^6} \approx \frac{1}{z^5} - \frac{1}{z^3}\frac{1}{3!} + \frac{1}{z}\frac{1}{5!} \dots$

(e) It's always possible to compute the residue of a function at its singular point using the standard residue formula.

FALSE

To compute residues of essential singularities you have to use Laurent Series sometimes, e.g. $e^{1/z}$ at $z=0$.