Complex Analysis

Math 214 Spring 2014 | Fowler 307 MWF 3:00pm - 3:55pm

 | http://faculty.oxy.edu/ron/math/312/14/

Class 28: **Wednesday April 16**

TITLE Introduction to Linear Fractional Transformations
CURRENT READING Saff & Snider, §7.2
HOMEWORK Saff & Snider, §7.2 14, 15, 21, **27***.

SUMMARY
We shall follow up on some odds and ends (Cauchy's Second Residue Theorem and Jordan's Lemma) and begin our in-depth look at the most useful mapping, the linear fractional transformation (LFT).

Cauchy's Second Residue Theorem

If a function $f(z)$ is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\oint f(z)\,dz = 2\pi i\,\mathbf{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\mathbf{;\ 0}\right]$$

In other words, instead of finding the residues of all the singularities of the given function $f(z)$ which lie inside the given contour C, all you need to do is find the residue at a single point, $z = 0$, of the associated function $\frac{1}{z^2}f\left(\frac{1}{z}\right)$. Note what's really going on involves finding the residue of the function at the point at infinity.

EXAMPLE

Evaluate $\oint_{|z|=2\pi} \tan(z)\,dz$ using Cauchy's Second Residue Theorem.

Evaluate $\oint_{|z|=2} \frac{e^{1/z}}{z-1}\,dz$ using Cauchy's Second Residue Theorem.

Exercise

Evaluate $\oint_{|z|=2} \frac{3z+2}{z^2+1}\,dz$ using Cauchy's Second Residue Theorem. (Confirm your answer by using Cauchy's (First) Residue Theorem.)

Recall The Properties of the Inversion Mapping

The mapping $w = 1/z$ maps the extended complex plane to itself on a one-to-one basis.

The mapping $w = \dfrac{1}{z}$ transforms *circles and lines* into *circles and lines*

Lines passing thru the origin	$\longmapsto$	Lines passing thru the origin
Lines NOT passing thru the origin	$\longmapsto$	Circles passing thru the origin
Circles passing thru the origin	$\longmapsto$	Lines NOT passing thru the origin
Circles NOT passing thru the origin	$\longmapsto$	Circles NOT passing thru the origin

Bilinear Transformations

Consider transformations of the form

$$w = f(z) = \frac{az+b}{cz+d} \qquad \text{where } ad - bc \neq 0$$

They are also known as **linear fractional transformations** or **Möbius transformations**. It's easy to see that you can re-write this to produce an expression of the form

$$Azw + Bz + Cw + D = 0$$

where A, B, C and D can be expressed in terms of a, b, c and d.

Bilinear Transformation as composite mapping

Notice that if $c = 0$ then our bilinear transformation (linear in z and w) becomes just a linear transformation in z.

If $c \neq 0$ then we can re-write $w = f(z)$ as

$$w = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz+d}$$

To show this, all we have to do is remember polynomial division:

If we look at the linear fractional transformation this way, we can see that it can be written as a composition of two linear transformations and an inverse mapping.

$$w = cz + d, \qquad w_1 = \frac{1}{w}, \qquad w_2 = \frac{a}{c} + \frac{bc - ad}{c}w_1$$

Find the composition of these three mappings above, so that $w_2 = T(z)$, and by so doing, show that T is a "LFT."

Thus LFTs can be thought of as a ____________________ followed by a ____________________ followed by a ____________________.

Therefore, we know that LFT's map *circles* and *lines* to ____________________ and ____________________

Properties of Linear Fractional Transformations

Let f be a Möbius transformation. Then

- f can be expressed as the composition of a finite number of rotations, translations, magnifications and inversions
- f maps the *extended complex-plane* to itself
- f maps the class of circles and lines to circles and lines
- f is **conformal** (i.e. $f'(z) \neq 0$) at every point besides its pole

Poles and Fixed Points

A **pole** (regular singularity) of a function is a point z_0 where $\lim_{z \to z_0} f(z) = \infty$

A **fixed point** of a function $f(z)$ is a point z_0 such that $f(z_0) = z_0$. That is, a fixed point in the z-plane gets mapped to the same spot in the w-plane.

Find the poles of $T(z) = \dfrac{az+b}{cz+d}$. How many poles does it have? How many fixed points does it have? (These answers should depend on a, b, c and d.)

If a line or circle passes thru the pole of T then it must be mapped to a shape that goes thru the point at infinity. **Why?** What kind of shape would do that?

So, if a line or circle does NOT pass thru the pole of T it must get mapped to what kind of shape?

Where does T map the point at infinity?

`THEOREM:` **Circle-Preserving Properts of LFTs**

If C is a circle in the z-plane and if $w = T(z) = \frac{az+b}{cz+d}$ is an LFT, then the image of C under T is either a circle or a line in the extended complex w-plane. The image is a line if and only if $c \neq 0$ and the pole of T, $z = -d/c$ is on the circle C.

Question Does the function $w = \frac{2z-1}{iz+1}$ map the circle $|z - i| = 1$ to a circle or a line?

Answer ____________________

Inverses of LFTs

Since T is a one-to-one mapping on the extended complex plane, it has an inverse. If you solve $w = T(z)$ so that $z = T^{-1}(w)$, then

$$T^{-1}(w) = \frac{-dw+b}{cw-a}, \qquad (ad - bc \neq 0)$$

Note that T^{-1} is also an LFT. So, the inverse of an LFT is another LFT. But, wait, there's more! In general, if S and T are two LFTs, then $S(T(z))$ is also an LFT, i.e. the composition of two LFTs is also an LFT.

EXAMPLE

Find the image of the *interior* of the circle $C : |z - 2| = 2$ under the LFT given by $w = f(z) = \dfrac{z}{2z - 8}$ Sketch the image and pre-image of C under $w = f(z)$

Exercise

Show that the image of the unit disk $|z| \leq 1$ under the mapping $w = i\dfrac{z-1}{z+1}$ is the set $\{w \in \mathbb{C} : \mathrm{Im}(w) \leq 0\}$