Complex Analysis

Math 214 Spring 2014 — Fowler 307 MWF 3:00pm - 3:55pm

 — http://faculty.oxy.edu/ron/math/312/14/

Class 19: **Friday March 21**

TITLE The Cauchy-Goursat Theorem
CURRENT READING Zill & Shanahan, §5.3
HOMEWORK Zill & Shanahan, §5.3 2, 9, 12, 20, 25, 27. **23*,29***. §5.4 1, 8, 18, 22. **25***

SUMMARY
We shall be introduced to one of THE most important theorems in complex analysis, the Cauchy-Goursat Theorem (and also learn about the related Path Independence Theorem and the Deformation Invariance Theorem).

RECALL
We evaluated $\int_C 2\bar{z}^2 \, dz$ on different contours and obtained different values. This doesn't seem like it should be a surprise. Different paths, different contour integrals, right?

EXAMPLE

Let's evaluate $\int_C 2z^2 \, dz$ using the following contours:
(Sketch the contour and evaluate the integral.)
(i) C is the directed line segment from $z = 2$ to $z = -2$

(ii) C is the circular arc going from $z = 2$ to $z = -2$ (counter-clockwise).

(iii) C is the circular arc going from $z = 2$ to $z = -2$ (clockwise).

QUESTIONS TO DISCUSS

What do we notice about our values of $\int_C 2z^2\,dz$ using different contours this time?
What's the difference between the integral $\int_{-2}^{2} 2\overline{z}^2\ dz$ and $\int_{-2}^{2} 2z^2\ dz$?
Does the value of a contour integral depend on the path taken in each case?
Does path dependence of a contour integral depend on the function involved?
What property of the function is involved?

RECALL

We previously showed the value of $\displaystyle\int_C 2\overline{z}^2\ dz$ **did** depend on the contour used.

However the value of $\displaystyle\int_C 2z^2\ dz$ **did not** depend on the contour chosen and in each case the integral is equal to $\dfrac{-32}{3}$

How is $f(z) = 2z^2$ a different function than $g(z) = 2\overline{z}^2$?

$2z^2$ is ________________________ while $2\overline{z}^2$ is ________________________.

EXAMPLE

$\displaystyle\int_C 2z^2\ dz =$ where C is **any** contour from 2 to -2

Analyticity of Integrand $\Rightarrow$ Path Independence Of Integral

Fundamental Theorem of Contour Integration

Suppose that the function $f(z)$ is continuous in a domain (an open connected set) D and has an antiderivative $F(z)$ throughout D, i.e. $dF/dz = f(z)$ at each point in D. Then for *every* contour Γ lying in D connecting z_1 to z_2 we have the result

$$\int_\Gamma f(z)\,dz = F(z_2) - F(z_1)$$

In otherwords, the value of the integral of an analytic function is **independent of the path** chosen to connect z_1 and z_2!

Tpo repeat: The differeence between $f(z) = 2z^2$ and $g(z) = 2\overline{z}^2$ is we know that $2z^2$ has the property that it is continuous and is equal to the derivative of $2z^3/3$ on the open set $z \in \mathbb{C}$. $g(z)$ has no similar antiderivative. (This is not surprising, since $g(z)$ is not analytic and is only differentiable at $z = 0$ only while $f(z)$ is analytic everywhere.)

Note that an implication from this theorem is that the function $F(z)$ will be **analytic** and **continuous** on the domain D (since it has a derivative at every point of the open set D). We also know that $f(z)$ is analytic because of the restated version of the FTCI:

THEOREM: **Path Independence Theorem**

If a function $f(z)$ is analytic in a simply connected domain D and Γ is **any** contour lying in D then the value of $\displaystyle\int_\Gamma f(z)\,dz$ is independent of the path Γ.

A corollary of the above theorem is the even more famous **Cauchy-Goursat Theorem:**

THEOREM: **Cauchy-Goursat Theorem**

If $f(z)$ is analytic at all points interior to and on any simple closed contour Γ, then

$$\oint_{\Gamma} f(z)dz = 0.$$

You can think of this as a direct result of using the path independence idea and assuming the initial point and terminal point are identical, i.e. $z_1 = z_2$.

Restatement of Cauchy-Goursat Theorem

If a complex function $f(z)$ is analytic in a simply connected domain D then for **every** simple closed contour C lying in D, $\oint_C f(z)dz = 0$.

NOTE

Another word for "simple closed contour" is **loop**. Basically the two statements of the theorem are the same because the interior of a loop is a simply connected domain. The implications are enormous for how flexible we can be in deciding what contour to use to evaluate contour integrals. It basically means that when we are integrating functions which are analytic through most points in a domain we can change the contour we are given for a simpler one to deal with and know we will not be changing the value of the integral.

THEOREM: **Deformation Invariance Theorem**

Let f be an analytic function in a domain D containing loops Γ_0 and Γ_1. If these loops can be continuously deformed into each other by passing through points only in D, then

$$\oint_{\Gamma_0} f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz$$

Simply Connected and Multiply Connected Domains

A **simply connected** domain D is one such that *every* simple closed contour (i.e. loop) lying in D encloses only points of D A domain which is *not* simply connected, is called **multiply connected**.

GROUPWORK

Sketch and classify the following domains as **simply connected** or **multiply connected**

$\mathcal{A} = \{z \in \mathbb{C} : 1 < |z| < 2\}$ $\mathcal{B} = \{z \in \mathbb{C} \setminus \{\ \mathrm{Re}(z) < 0 \cap\ \mathrm{Im}(z) = 0\}\}$

$\mathcal{C} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < 1\}$ $\mathcal{D} = \{z \in \mathbb{C} : |z| < 4 \setminus \{|z - i| < \frac{1}{2}\ \cup\ |z + i| < \frac{1}{2}\}\}$

`FACT`: Simply connected domains have the property that every loop in D can be continuously deformed in D to a single point.

`FACT`: $\int_C f(z)dz \equiv 0$ if C is a point.

Exercise

Evaluate the integral I below, where Γ is SOME RANDOMLY SHAPED, **simple, closed** (positively oriented) contour.

$$I = \oint_\Gamma \frac{dz}{z - z_0}$$

Does it matter where $z_0 \in \mathbb{C}$ is located to determine the value of I?

EXAMPLE

Evaluate $\displaystyle\int_C \frac{5z+7}{z^2+2z-3}\, dz =$ where C is the circle $|z-2| = 2$ traversed clockwise.

GROUPWORK

Evaluate $\displaystyle\int_C \frac{3z-2}{z^2-z}\, dz =$ where C is the "figure eight" contour shown below