Complex Analysis

Math 214 Spring 2014 Fowler 307 MWF 3:00pm - 3:55pm

 http://faculty.oxy.edu/ron/math/312/14/

Class 12: **Wednesday February 19**

TITLE Applications of Harmonic Functions
CURRENT READING Zill & Shanahan, Section 3.5.
HOMEWORK Zill & Shanahan, §3.4 #6, 11, **14***; §3.5 #7, 12, **16***;

SUMMARY
We shall begin looking at applications of analytic functions by taking a closer look at harmonic functions and situations in which they can be useful.

The Complex Velocity Potential

In Fluid Dynamics, the complex velocity potential is a useful quantity used to analyze certain fluid fields. It can be defined as

$$\Phi(z) = \phi(x, y) + i\psi(x, y)$$

where $\phi(x, y)$ is called the *velocity potential* and $\psi(x, y)$ is called the *stream function*. Potential functions are useful because simply by taking the proper partial derivative one can obtain the velocity components.

One can compute expressions for V_x (horizontal component) and V_y (vertical component) of the velocity of a fluid, denoted by $\vec{v}$ from the complex velocity potential:

$$\overline{\frac{\partial \Phi}{\partial z}} = V_x + iV_y = \frac{\partial \phi}{\partial x} - i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$$

Note that $\vec{v} = \begin{bmatrix} V_x \\ V_y \end{bmatrix}$

Exercise

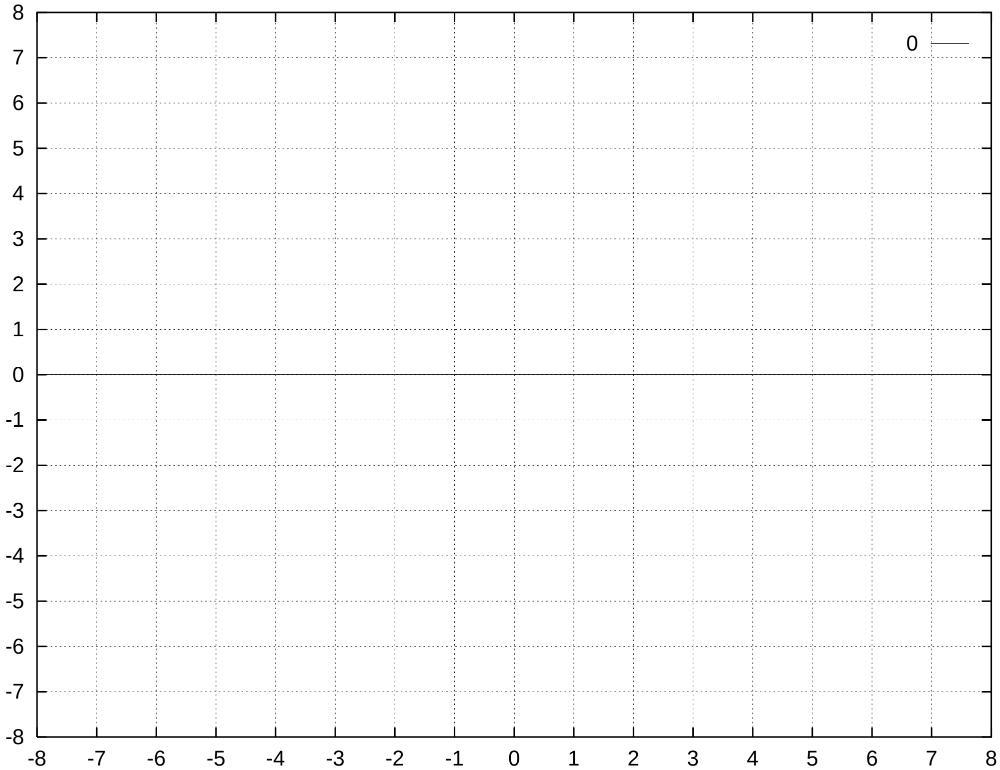
Given $\Phi(z) = z^2$ compute the stream function ψ and velocity potential ϕ and obtain expressions for fluid velocity $\vec{v}(x, y)$

Streamlines and Equipotentials

Recall that the level curves of a function $f(x, y)$ occur when $f(x, y) = \text{constant}$. Level curves of $\phi(x, y)$ are called **equipotentials** and level curves of $\psi(x, y)$ are called **streamlines**. They have particularly interesting physical meaning.

GroupWork

Sketch the streamlines and equipotentials relating to the flow described by $\Phi(z) = z^2$ on the grid below. In other words, sketch $\phi(x, y) = c$ and $\psi(x, y) = d$, where c and d are $\pm 1, \pm 2$, et cetera. What kind of curves are they?



If you look carefully (or if you sketched accurately) the equipotentials and streamlines intersect at right angles. This is not an accident. Level curves for the real and imaginary parts of an analytic function $f(z)$ are always **orthogonal**.

We can show this by remembering the meaning of the gradient of a function $f(x, y)$, denoted by ∇f, the dot product and applying the Cauchy-Riemann equations:

Exercise

Show that the level curves of harmonic conjugates of an analytic function always intersect perpendicularly.

The Dirichlet Problem

A problem where one is looking for a function $\phi(x, y)$ which satisfies a partial differential equation (like Laplace's Equation) in an open connected set D (i.e. a domain) and which equals equals a known function $g(x, y)$ along the boundary of D (sometimes represented by ∂D) is called a **Dirichlet problem**.

EXAMPLE

Given the following Dirichlet problem for Laplace's Equation

$$\phi_{xx} + \phi_{yy} = 0 \qquad x_0 < x < x_1, \qquad -\infty < y < \infty$$

$$\phi(x_0, y) = k_0, \quad \phi(x_1, y) = k_1, \qquad -\infty < y < \infty$$

show that the function

$$\phi(x, y) = \frac{k_1 - k_0}{x_1 - x_0}(x - x_0) + k_0$$

is a solution of the given Diricjhlet problem.

GroupWork

Adapted from **Zill & Shanahan, page 146, #11**. Let $D = \{z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1\}$. Given the boundary conditions are $\phi(0, y) = 50$ and $\phi(1, y) = 0$ find **(a)** the solution $\phi(x, y)$ to the corresponding Dirichlet problem on D for Laplace's equation and **(b)** the corresponding complex potential $\Phi(z)$.