Complex Analysis

Math 214 Spring 2014 — Fowler 307 MWF 3:00pm - 3:55pm

 — http://faculty.oxy.edu/ron/math/312/14/

Class 9: Monday February 10

TITLE Limits and Continuity of Complex Functions
CURRENT READING Zill & Shanahan, Section 3.1
HOMEWORK Zill & Shanahan, §3.1.1: #2, 11, 17, **20***; §3.1.2: #28, 31, 37, **50***;

SUMMARY
We shall formally define the definition of the limit of a complex function to a point and use this definition to define the concept of **continuity** in the onctext of a complex function of a complex variable.

Limits

Suppose that $f(z)$ is defined on a deleted neighborhood of $z_0 \in \mathbb{C}$. In order to say that $\lim_{z \to z_0} f(z) = w_0$ we must be able to show that

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

This may look like dense mathematical language, but in english this means that for every positive number ϵ (no matter how small) there exists a number δ (which depends on the choice of ϵ) so that regardless of how close you get to the point z_0 in the deleted neighborhood around it in the z-plane you can also get arbitrarily close to the value w_0 in the w-plane.

Visualize The Limit

Let's try and prove the result $\lim_{z \to i} z^2 = -1$ (See Example 2 on page 100 of Zill & Shanahan)

Nonexistence of a Complex Limit

If $f(z)$ approaches two complex numbers $L_1 \neq L_2$ along two different paths towards z_0 then $\lim_{z \to z_0} f(z)$ does not exist.

Exercise

Show that $\lim_{z \to 0} \frac{z}{\overline{z}}$ does not exist. (HINT: pick a vertical path and a horizontal path)

Rules for Limits

The rules on limits of complex functions are identical to the rules for limits of real functions of real variables (as you'd expect)

Suppose that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$ then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0 \tag{1}$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0 \tag{2}$$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \qquad (W_0 \neq 0) \tag{3}$$

$$\lim_{z \to z_0} |f(z)| = |w_0| \tag{4}$$

$$\lim_{z \to z_0} c = c \tag{5}$$

$$\lim_{z \to z_0} z^n = z_0^n \tag{6}$$

$$\lim_{z \to z_0} P(z) = P(z_0) \qquad \text{(where } P(z) \text{ is a polynomial)} \tag{7}$$

`THEOREM`

Given $f(z) = u(x, y) + iv(x, y)$, $\quad z_0 = x_0 + iy_0 \quad$ and $w_0 = u_0 + iv_0$

$\lim_{z \to z_0} f(z) = w_0$ **if and only if** $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$

GROUPWORK

Use the above properties to evaluate the following limits (note which properties you use).

(a) $\lim_{z \to 1+2i} 2|z| + iz^2 + 2.5 - i =$

(b) $\lim_{z \to 3\pi i} ze^z =$

(c) $\lim_{z \to 0} \frac{z^8 + z^4 + z^2 + z - 1}{z^3 + 4z^3 - 9} =$

(d) $\lim_{z \to 1-i} 2xy - ix^2 + iy^2 =$

Continuity

A complex function $f(z)$ is **continuous** at a point z_0 if *all three* of the following statements are true

1: $\lim_{z \to z_0} f(z)$ exists

2: $f(z_0)$ exists

3: $\lim_{z \to z_0} f(z) = f(z_0)$

Consider the function below:

$$f(z) = \begin{cases} \dfrac{z^2+4}{z-2i}, & z \neq 2i \\ \\ 3+4i, & z = 2i \end{cases}$$

Answer the following questions

1. What is the value of $\lim_{z \to 2i} f(z)$?

2. Is $f(z)$ continuous at $z = 2i$?

3. Is $f(z)$ continuous at points $z \neq 2i$?

We say that the function $f(z)$ defined above has a **removable singularity** at $z = 2i$.

Exercise

Write down a definition of $f(z)$ which is continuous, i.e. a $f(z)$ which has had the singularity removed.

More Aspects of Continuity

As with real functions of a real variable, **sums**, **differences**, **products** and **compositions** of continuous functions are continuous.

`THEOREM`

Complex polynomial functions are continuous on the entire complex plane. Functions with this property are often called **entire** functions.

`THEOREM`

$f(z)$ continuous $\Longleftrightarrow$ $u(x,y)$ and $v(x,y)$ continuous.

`THEOREM`

When $f(z)$ continuous in a region R, then $|f(z)|$ is also continuous in the region R and if R is a *bounded* and *closed* set (i.e. it is **compact**) then there exists a positive number M so that $|f(z)| \leq M \quad \forall z \ni R$.

Introduction to Branch Cuts and Branch points

Consider the principal square root function $z^{1/2}$. This function is defined as $\sqrt{|z|}\exp\left(\frac{i\mathrm{Arg}(z)}{2}\right)$. It has as its domain the set $\mathcal{D} = \{z \in \mathbb{C} : \mathbb{C} \setminus \{0\}\}$.

EXAMPLE

Show that the principal square root function is not continuous at $z = 1$.

If we remove the set of points along the negative real axis from $\mathcal{D}$ (the domain of the principal square root function) and define a new function $f_1(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $-\pi < \theta < \pi$ this new function is called the **principal branch** of the multiple-valued square root function $z^{1/2}$. The set of points we removed, $\{z \in \mathbb{C} : \mathrm{Im}(z) = 0 \cap \mathrm{Re}(z) \leq 0\}$, is called a **branch cut** of $z^{1/2}$.

Q: How is the principal branch $f_1(z)$ different from the principal square root function?
A: The principal branch is continuous on its entire domain.
Q: Can we deefine other branches of $z^{1/2}$?
A: Yes, we could define another branch of the principal square root function as $f_2(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $\pi < \theta < 3\pi$. And a third branch of $z^{1/2}$ as $f_3(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $0 < \theta < 2\pi$

Exercise

Show that $f_2(z) = -f_1(z)$ for all $|z| > 0$.

Branch of a Multivalued Function

A branch of a multi-valued function is a single-valued analogue which is continuous on its domain.

Branch Cut

The set of points that have to be removed from the domain of a multivalued function to produce a branch of the function.

Branch Point

The point in the complex plane which lies in every branch cut of a complex function. It is often the origin.