Complex Analysis

Math 214 Spring 2014 | Fowler 307 MWF 3:00pm - 3:55pm
 | http://faculty.oxy.edu/ron/math/312/14/

Class 4: Wednesday January 29

TITLE Polynomial Equations of a Complex Variable and Roots of Complex Numbers
READING Zill & Shanahan, Section 1.4 and 1.5
HOMEWORK Saff & Shanahan, §1.4 # 4,5,17,18,20 **Extra Credit: #29**

SUMMARY
We learn how to find the roots of a complex variable, which is necessary in obtaining solutions of polynomial equations of a complex variable.

UPDATE
The answers to Exercise 2 from Class #3 are:
A: $e^{i\pi} = \cos\pi + i\sin\pi = -1$
B: $1 + i = |1 + i|e^{i \text{ arg } (1+i)} = \sqrt{2}e^{\frac{i\pi}{4}+2ni\pi}, \quad n \in \mathbb{Z}$
C: $(1-i)^5 = \left(|1-i|e^{i \text{ Arg } (1-i)}\right)^5 = \left(\sqrt{2}e^{i\frac{-\pi}{4}}\right)^5 2^{5/2}e^{\frac{-i5\pi}{4}} = \sqrt{32}e^{\frac{3\pi i}{4}} = \sqrt{32}\frac{(i-1)}{\sqrt{2}} = \sqrt{16}(-1+i) = -4+4i$

We now know how to deal with real integer powers of complex numbers in a nice way (by using DeMoivre's Formula and exponential form).

Fractional Exponents

What about real fractional powers of complex numbers, i.e. roots? That is, we want to solve an equation like

$$z^n - z_0 = 0 \tag{1}$$

where z_0 is a known complex number, and we are trying to find the corresponding value(s) of $z = z_0^{1/n}$ which solve this equation.
Suppose we write z_0, z and z^n in polar form:

$$z_0 = \tag{2}$$
$$z = \tag{3}$$
$$z^n = \tag{4}$$

where $|z_0| = r_0$, $|z| = r$, $\text{Arg } z_0 = \theta_0$ and $\text{Arg } z = \theta$
We can rewrite (1) as $z^n = z_0$ and $z = z_0^{1/n}$. Then we can obtain expressions for the r and θ that correspond to this $z = re^{i\theta}$.

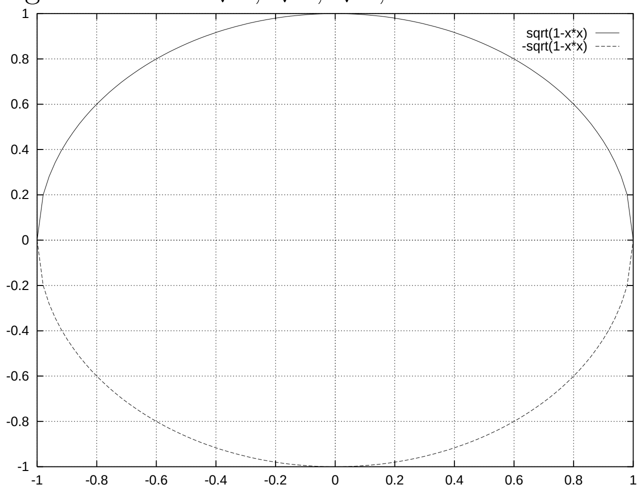
Exercise 1
Show that the expressions for r and θ in terms of r_0 and θ_0 are $r = {r_0}^{1/n}$ and $\theta = \dfrac{\theta_0 + 2k\pi}{n}, k \in \mathbb{Z}$

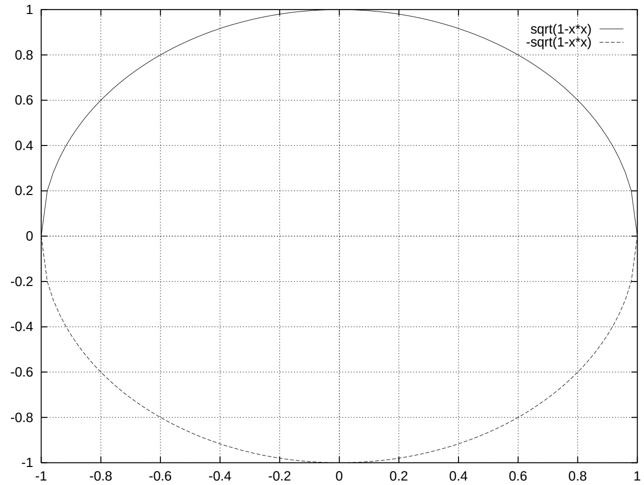
Roots of Unity

We are interested in finding the n^{th} roots of unty, i.e. z such that

$$z^n = 1, \qquad n = 1, 2, 3 \ldots$$

On the following axes, draw vector representations of the n^{th} *roots of unity* when $n = 2$, $n = 3$ or $n = 4$. How many distinct solutions to $z^n = 1$ are there? In other words, we are trying to evaluate $\sqrt{1}$, $\sqrt[3]{1}$, $\sqrt[4]{1}$, ...





What do you think the 5^{th} roots of unity will look like?

EXAMPLE 1

Compute the solutions to the equation $z^5 = i$ and write them in polar (and rectangular) form. Sketch these solutions on the grid on the right.

Using DeMoivre's Formula and the result on n^{th} roots we can obtain a general formula for evaluating $z^{m/n}$

$$z^{m/n} = c_k = |z|^{m/n} \exp\left(\frac{mi(\theta + 2\pi k)}{n}\right), \qquad k = 0, 1, \ldots, n-1, \quad \text{where } \theta = \text{ Arg } z$$

These n roots can be written as

$$c, c\omega_n, c\omega_n^2, c\omega_n^3, \cdots, c\omega_n^{n-1}$$

where c is *any* n^{th} root of a non-zero complex number, and $\omega_n = \exp\left(\frac{2i\pi}{n}\right)$

GROUPWORK

(1) Prove that $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ (use DeMoivre's Formula)

(2) Evaluate $\sqrt{5-12i}$

(3) Solve $w^3 - i = -\sqrt{3}$

(4) Solve $w^{4/3} + 2i = 0$