Hector Lopez

Complex Analysis Research Paper

Due: 4/18/14

Development of the CREs: From D'Alembert to Cauchy

During the 19th century, mathematicians founded a new branch of mathematics in order to explore the problems concerning imaginary numbers. Before this era, mathematicians considered imaginary numbers (i.e. $\sqrt{-1}$) as useless and irrelevant to the subject of mathematics, for such numbers were poorly understood. The investigation of the imaginary occurred after the 18th century when mathematicians began applying certain concepts to complex numbers. For example, in 1777, Leonhard Euler made the symbol i stand for $\sqrt{-1}$ and determined the formula $x + iy = r(cos\theta + isin\theta)$. Derivations of certain theorems and ideas can have a straightforward history, but the Cauchy- Riemann Equations has an intricate past, as its derivation is found in proofs of different mathematicians. These mathematicians include Jean le Rond D'Alembert, Leonhard Euler, Augustin- Louis Cauchy, and Bernhard Riemann.

Before highlighting the history of the system of equations, its definition should be noted. Theorem 3.3.1 states, "Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first- order partial derivatives of u and v exist and satisfy the Cauchy- Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$" (Shanahan, 131). This theorem, which presents the two partial differential equations, can be utilized to determine the analyticity of a function. A function $f(z)$ is labeled as analytic if the CREs are satisfied and if the function "is differentiable at z_o and at every point in a neighborhood surrounding z_o"

(Buckmire, W.S. 11). Therefore, if the partial differential equations are continuous at $z_o = x_o + iy_o$, then the function $f(z)$ is differentiable at z_o, and the derivative is denoted as

$$f'(z) = f'(x_o + iy_o) = u_x(x_o, y_o) + iv_x(x_o, y_o) = v_y(x_o, y_o) - iu_y(x_o, y_o).$$

Hence, the system of equations is useful in determining differentiability on a domain D.

Despite of being referred to as the Cauchy- Riemann equations, Jean Le Rond D'Alembert unknowingly founded the system of equations. It is important to note his area of interest before stating where he first presented the system of equations. He, making important contributions to the subject of mathematics, focused on the investigation of physical laws in mathematical terms. Working along with Leonhard Euler, D'Alembert was able to contribute to the growth of mathematical physics, so it is not surprising that D'Alembert obtained the partial differential equations in the area of physics, specifically hydrodynamics.

In his work, *An Essay on a New Theory of Fluid Resistance,* Jean le Rond D'Alembert presented promising ideas about fluid mechanics. He introduced the concept of velocity fields, and unknowingly, he derived a system of equations that were similar to the CREs. In Sect. III, §§ 57-60 of his essay, D'Alembert explained his attempt of finding a formula that gave the velocity of a fluid under certain conditions. "The velocity components along the axes *x* and *y* are proportional to unknown functions *p* and *q*, respectively" (Bottazzini and Gray, 86). His hypothesis was

$$dq = Mdx + Ndz \text{ and } dp = Ndx - Mdz$$

where each partial can be simplified to

$$(M + iN)(dx - idz) \text{ and } (M - iN)(dx + idz)$$

He made $du = dx - idz$ and $dt = dx + idz$, $M + iN = \alpha$ and $M - iN = \beta$. He determined that α must be a function of u, indicating that $M + iN$ must be a function of $F + x - iz$. Likewise, he concluded that β was a function of t, meaning that $M - iN$ is a function of $G + x + iz$. Of course, this is not his entire proof, but only a portion that's relatable to the partial differential equations. Because of his hypothesis, he was able to obtain the partial differential equations $\frac{\partial p}{\partial z} = -\frac{\partial q}{\partial x}$ and $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial z}$, which resemble the Cauchy Riemann Equations. D'Alembert, unfortunately, did not continue using these partial differential equations in his essay, thus "he did not connect [the equations] with any development of complex number theory" (Bottazzini & Gray, 86).

During the 18th century, the interest in complex analysis accelerated. It was during this period that Leonhard Euler attempted to manipulate D'Alembert's partial differential equations. "In dealing with plane fluids, Euler needed the differentials $udx + vdy$ and $udy - vdx$ to be integrable" (Bottazini & Gray, 88). Using a method similar to D'Alembert's, Euler used the linear combinations

$$u(dx + idy) - iv(dx + idy) \text{ and } u(dx - idy) + iv(dx - idy),$$

Becoming

$$(u - iv)(dx + idy) \text{ and } (u + iv)(dx - idy)$$

Euler remarked that the conditions are satisfied if $u - iv$ is any function of $x + iy$ and if $u + iv$ is any function of $x - iy$. Euler added that the functions are represented by real values u and v, and the "imaginary be destroyed". Decisively, "Euler established that the equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ is the condition under which $vdx + udy$ is the exact differential of some function"(Brating, 5). Here, Euler is establishing certain conditions for the partial differential equation that will define differentiability of complex functions. Like D'Alembert, Euler did not continue working with the partial differential equations. This is why Euler and D'Alembert were not credited with the derivation of the system of equations because they did not give a conclusive definition or meaning to the equations.

The creation of the CRE's occurred in 1818 when Augustin- Louis Cauchy published a short paper on the integration of differential equations. In this paper, he "observed that… $Pdy - Qdx$ and $Pdx + Qdy$ will be complete differentials if both P and Q denote two real functions x and y related by an equation of the form $\emptyset(x + yi) = P - Qi$"(Bottazzini & Gray, 102). This led to the hypothesis

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial y} i = i\emptyset'(x + iy) = \frac{\partial P}{\partial x} i + \frac{\partial Q}{\partial x}$$

Hence,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y}$$

In the case where $y = M + Ni,\ u = M(x, z)$ while $v = N(x, z)$. However, Cauchy failed to express his results in geometrical terms, thus questioning the legitimacy of his results. "There is some evidence that in 1814 Cauchy did not yet have a clear understanding of what was going on" (Bottazzini &Gray, 102). Even though the thinking process behind this "Cauchy Theorem" is questioned, we still employ his theorem. Sylvestre- Francois Lacrois, an affluent mathematician, and Andre- Marie Legendre, another important mathematician, agreed that Cauchy's use of the imaginary obeyed the rules of analysis, establishing the legitimacy of Cauchy's Theorem. Because of Cauchy, the central aspect of the partial differential equations has been implemented.

In his dissertation on the theory of functions, Bernhard Riemann applied his knowledge of analyticity to complete the system of equations. Riemann stated, "A complex variable quantity w is called a function of another complex variable quantity z when the one changes with the other in such a way that the value of the derivative $\frac{dw}{dz}$ is independent of the value of the differential dz" (1851, p.5). This is similar to the modern definition of an analytic function where w is differentiable at z_o and at every point in the neighborhood of z_o. This implies that a function is analytic if it is differentiable and continuous on the domain *D.* Both conditions, differentiability and continuity, are required in order for a function to be referred to as analytic. Also, if a function is differentiable, it does not necessarily indicate that the function is analytic. This is proven by that fact that differentiability does not imply continuity. Therefore, in order to see if the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic, the Cauchy's Theorem is applied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

As stressed in Buckmire's *Worksheet 11,*

$$ANALYTICITY \leftrightarrow C.R.E + Continuity\ u_x, u_y, v_x, v_y$$

"Riemann remarked that these equations can be used to study the individual properties of u and v and thus the complex function $w = u + iv$" (Gray, 22). More generally, by following the Cauchy- Riemann conditions, one can find the harmonic conjugate of $u(x, y)$ or $v(x, y)$. Cauchy and Riemann were able to institute the terms for this system of equations, and the final definition is stated in Theorem 3.3.1.

In their works, Jean le Rond D'Alembert and Leonhard Euler had aspects that resembled the Cauchy- Riemann equations. "Nevertheless, neither man carried out further investigations of the properties of complex functions: instead, they limited themselves to the study of the real and the complex parts of such functions" (Grattan-Guinness, 420). This explains why Cauchy and Riemann are credited for the development of the equations. Even though, the equations are linked to other mathematicians like D'Alembert and Euler, the main creators are evidently Cauchy and Riemann.

Works Cited:

Bottazzini, Umberto, and Jeremy Gray. *Hidden Harmony—Geometric Fantasies: The Rise of Complex Function Theory*. N.p.: Springer, 2013. N. pag. Web. 10 Apr. 2014.

Brating, Kajsa. *Malmsten's Proof of The Integral Theorem -an Early Swedish Paper on Complex Analysis*. Rep. no. 1. N.p.: Uppsala U, 2002. Web. 11 Apr. 2014.

Grattan-Guinness, I. *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*. London: Routledge, 1994. Print.

Gray, Jeremy. *Linear Differential Equations and Group Theory from Riemann to Poincare*. Boston: Birkhaüser, 1985. Print.