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Complex Analysis Project

Solving for the Roots of the Cubic Equation

Finding the solution to the roots of a polynomial equation has been a fundamental problem of mathematics for centuries. As mathematicians, we all know how to get the solution to the roots of a polynomial of degree two, which is given by the quadratic formula that depends upon the coefficients of x^2, x and the constant term c. There is an explicit formula for discovering the roots of a quadratic equation, but is there an explicit formula for higher order polynomials, such as the cubic equation? As it turns out, an explicit formula is given that produces a solution to the roots of a cubic polynomial, known as the cubic formula, which was first discovered in the 16th century and later revealed to the public by the mathematician Gerolamo Cardano in his book Ars Magna, or better known as The Great Art. In his work, Cardano uses geometric figures to develop his method in arriving to the cubic formula, which thereby solves the cubic equation, finding both real and imaginary roots of the equation. Cardano's method of solving for the general cubic equation involves reducing the equation

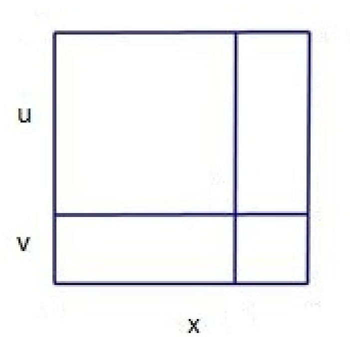
$$z^3 + az^2 + bz + c = 0 \tag{1}$$

to a depressed cubic equation through a translation of z, which allows us to geometrically derive a solution for the roots. We will also be testing this cubic formula on a set of coefficients for a, b, and c by finding a depressed cubic equation and using the cubic formula to find its roots. Then, we will graph the original polynomial and depressed equation to compare x-intercepts, and find the final solutions of the cubic equation.

To find a root of the cubic equation, it is sufficient to find a depressed cubic equation by a means of translation. Depressing a cubic equation means to find a linear formula that will be equal to the cube of the independent variable. In other words, the cubic equation would be depressed to the form

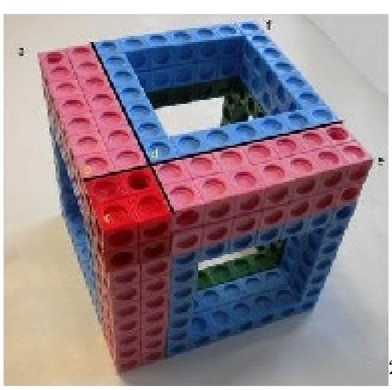
$$x^3 = mx + n \tag{2}$$

where m and n are constants, and x is a translation of z. In the case of Equation (1), this involves translating z into the form $z = x - a/3$, and plugging this equation for z into Equation (1). We will demonstrate how this transforms Equation (1) into the form of Equation (2) later. Cardano chose to solve for the cubic equation in this manner because at the time, there was no algebraic method for solving for the roots of the cubic equation. However, he could represent a cubic such as x^3 geometrically as a cube with edges length x, and he could decompose the cube as to solve for coefficients m and n in Equation (2). Cardano began by dividing x into two shorter lengths u and v, such that $x = u + v$, which in turn divides each face of the cube into four different rectangles, as shown below.



[1]

Consequently, this decomposes the cube into eight smaller cubes, as shown below.


[2]


[3]

[1,2,3]Images taken and/or modified from www.maa.org

These cube decompositions have volumes u^3, $3u^2v$, $3uv^2$, and v^3. Since the sum of the volumes of these eight cubes equals the volume of the original cube, Cardano could write x^3 explicitly as

$$x^3 = u^3 + 3u^2v + 3uv^2 + v^3 \tag{3}$$

which, amazingly, is the binomial expansion for $(u+v)^3$. Now that Cardano had an explicit formula for x^3, he could form it into a linear equation by letting $m = 3uv$ and $n = u^3 + v^3$. Hence,

$$x^3 = m(u+v) + n = mx + n$$

Once Cardano had accomplished this, he needed to find a formula for x in terms of m and n that would solve Equation (2), which he could achieve since he had formulas for m and n in terms of u and v. Since x is equivalent to $(u^3)^{\frac{1}{3}} + (v^3)^{\frac{1}{3}}$, Cardano first solved for u^3 and v^3 by using the procedure as follows:

$$\begin{gathered} u^3 = n - v^3 = n - (\tfrac{m}{3u})^3 \\ u^3 = n - \tfrac{m^3}{27u^3} \\ 27(u^3)^2 = 27nu^3 - m^3 \\ 27(u^3)^2 - 27nu^3 + m^3 = 0 \end{gathered}$$

Since this is a quadratic equation and the quadratic formula had already been established, Cardano used the quadratic formula to solve for u^3.

$$\begin{aligned} u^3 &= \frac{27n \pm \sqrt{27^2n^2 - 4(27)m^3}}{2(27)} \\ &= \frac{n}{2} \pm \left(\frac{27^2n^2 - 4(27)m^3}{4(27)^2}\right)^{\frac{1}{2}} \\ &= \frac{n}{2} \pm \left(\frac{n^2}{4} - \frac{m^3}{27}\right)^{\frac{1}{2}} \end{aligned}$$

Using this same process, the same solution as above is given for v^3. However, there is the condition that $n = u^3 + v^3$ and n must be a real number, which means that u^3 and v^3 must be complex conjugates of one another. Therefore, the explicit solution to the depressed cubic equation in terms of the coefficients m and n is given by:

$$x = \left[\frac{n}{2} + \left(\frac{n^2}{4} - \frac{m^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{n}{2} - \left(\frac{n^2}{4} - \frac{m^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \tag{4}$$

Thus, Cardano arrived a solution to the depressed cubic equation given in Equation (2). To find the roots of the general cubic equation given in Equation (1), one simply needs to plug the above formula into $z = x + 1$. Now that we have found a formula which produces a root of a cubic equation, we will test it on an example of a cubic equation and compare the root found by this formula to the roots computed algebraically.

We will begin by transforming the general cubic equation into its depressed form as previously discussed by setting,

$$z = x - \frac{a}{3}. \tag{5}$$

Such that when we input this new value for z in Equation (1) we should get an output of Equation (2).

$$\begin{aligned} z^3 + az^2 + bz + c &= 0 \\ \left(x - \frac{a}{3}\right)^3 + a\left(x - \frac{a}{3}\right)^2 + b\left(x - \frac{a}{3}\right) + c &= 0 \\ x^3 - ax^2 + \frac{a^2}{3}x - \frac{a^3}{27} + ax^2 - \frac{2a^2}{3}x + \frac{a^3}{9} + bx - \frac{ba}{3} + c &= 0 \end{aligned}$$

After simplifying the above expanded equation and isolating the x^3 term to one side of the eqaution, we get the following depressed cubic equation:

$$x^3 = \left(\frac{a^2}{3} - b\right)x + \left(-\frac{2a^3}{27} + \frac{ba}{3} - c\right) \tag{6}$$

As we can see from Equation (6), our m and n-values of the depressed cubic equation are

$\left(\frac{a^2}{3} - b\right)$ and $\left(-\frac{2a^3}{27} + \frac{ba}{3} - c\right)$ respectively. In order to test Cardano's method we used our example cubic equation:

$$z^3 + 3z^2 - 3z - 9 = 0. \tag{7}$$

Where we have the (a, b, c) values equal to $(3, -3, -9)$, giving us through menial algebric computation our coefficient values for the depressed equation to be $m = 6$ and $n = 4$. When we plug-in these values into the general depressed cubic equation we get the depressed equation,

$$x^3 = 6x + 4 \tag{8}$$

for Equation (7). Now that we have our depressed cubic equation, we can plug in our m and n-values into the cubic formula for the general cubic equation:

$$\begin{aligned} x &= \left[\frac{n}{2} + \left(\frac{n^2}{4} - \frac{m^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{n}{2} - \left(\frac{n^2}{4} - \frac{m^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} \\ &= \left[\frac{4}{2} + \left(\frac{4^2}{4} - \frac{6^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[\frac{4}{2} - \left(\frac{4^2}{4} - \frac{6^3}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} \\ &= \left[2 + \sqrt{-4}\right]^{\frac{1}{3}} + \left[2 - \sqrt{-4}\right]^{\frac{1}{3}} \end{aligned}$$

When simplified further, we get a cubic root of:

$$x = [2 + 2i]^{\frac{1}{3}} + [2 - 2i]^{\frac{1}{3}} \tag{9}$$

In order to get a cubic root for our example cubic equation we use the corresponding coefficient value of our cubic equation and the cubic solution (9), and plug it into equation (5):

$$z = [2 + 2i]^{\frac{1}{3}} + [2 - 2i]^{\frac{1}{3}} - 1. \tag{10}$$

Using the online mathematical application Wolfram Alpha, we pluged-in (10) into our cubic equation (7) and got the expected value of zero. Thereby we were able to find a solution for our cubic example and thus proving that Cardano's method for calculating the cubic root of equation (1) works. Wolfram also provided us with the fact that the solution (10) is an alternate form of the cubed root of three, such that when solving equation (7) by using the method of factoring by grouping we get,

$$
\begin{aligned}
z^3 + 3z^2 - 3z - 9 &= 0 \\
z^2(z+3) - 3(z+3) &= 0 \\
(z^2 - 3)(z+3) &= 0.
\end{aligned}
$$

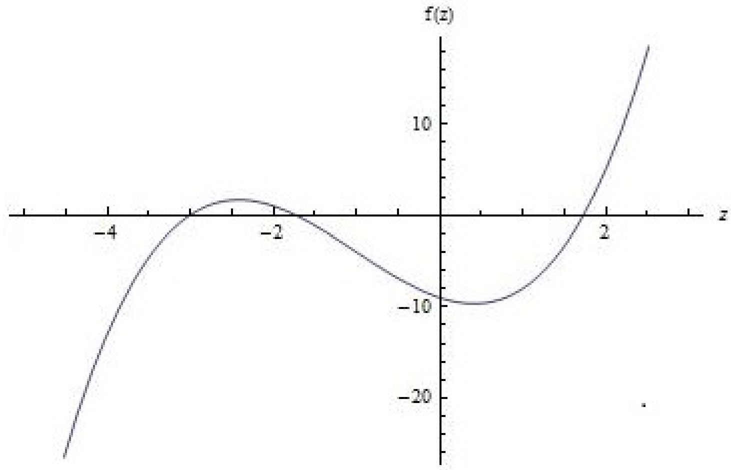
From this method we are able to conclude that the cubic roots to our cubic equation are $\pm\sqrt{3}$ and -3. We proceded to graph our polynomial given in equation (7) along with the polynomial from the depressed cubic equation (8) into Wolfram, therby estimating the x-intercepts from the graphs. In order to accomplished this we first manipulated equation (5), isolating x in terms of z with the a coefficent value equal to 3 given from equation (7):

$$
\begin{aligned}
x &= z + 1 \\
x^3 &= 6x + 4 \\
(z+1)^3 &= 6(z+1) + 4 \\
z^3 + 3z^2 + 3z + 1 &= 6z + 10.
\end{aligned}
$$

This implies that

$$z^3 + 3z^2 - 3z - 9 = 0.$$

Plugging in both of these polynomials into Wolfram produced the following graphs, where we can observe the corresponding blue line to the cubic polynomial and the purple line to the depressed cubic overlaying one another, thereby producing the same graph. The two graphs serve the purpose of demonstrating how the polynomials behave the same on the entire plane and proving the x-intercepts occuring at $\pm\sqrt{3}$ and -3 as expected from our algebraic results.



[4]

By deriving a formula that solves for the depressed cubic equation, Cardano provided another method to solve for the cubic equation in addition to the algebraic techniques such as the factorization by grouping. With both of these tools at our disposal, we are now guaranteed to be able to find at least one root of a cubic equation in the form of Equation (1). This method is especially useful for cubic equations that cannot be factored algebraically, as we now have a procedure from which we can find a root.

[4]Graph drawn through Mathematica

References

Branson, William B. ”Solving the Cubic with Cardano - Decomposing a Cube.” MAA. Mathematical Association of America, n.d. Web. 18 Apr. 2014. <http://www.maa.org/publications/periodicals/convergence/solving-the-cubic-with-cardano-decomposing-a-cube>.

”Cardano’s Derivation of the Cubic Formula.” Planetmath.org. N.p., n.d. Web. 18 Apr. 2014. <http://planetmath.org/cardanosderivationofthecubicformula>.

”Cardano’s Solution of the Cubic.” Algebra: The Search for an Elusive Formula. N.p., n.d. Web. 18 Apr. 2014. ¡http://sofia.nmsu.edu/ history/book/carda

Zill, Dennis G., and Patrick D. Shanahan. Complex Analysis: A First Course with Applications. 3rd ed. N.p.: Jones and Bartlett Learning, 2013. Print.