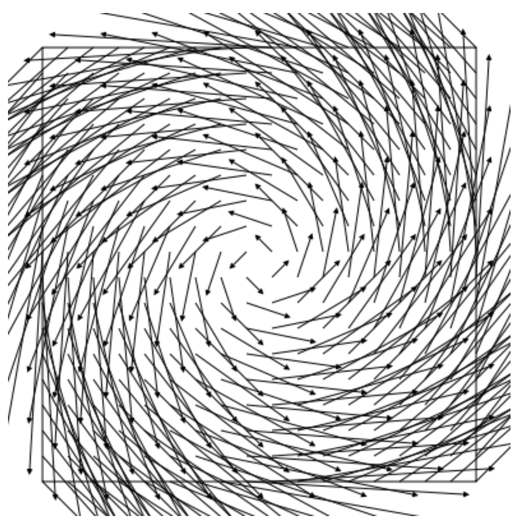
Representing Fluid Flows: Fields, Fluxes, and Circulation



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Spring 2014, Math312, Complex Analysis

OVERVIEW

This report explores how the physical properties of laminar fluid flows are "captured" by mathematical models. With the goal of bridging the gap between qualitative and computational understanding, we will focus on the two major characteristics of flow: flux and circulation. After establishing mathematical expressions for these concepts, we will apply them to an examination of ideal fluids and prove that flux and circulation over a closed loop in an ideal fluid are zero (problem adapted from Zill and Shanahan, Ch5.6, #31). Finally, we will consider how complex functions describing fluid flow may be utilized to solve for flux and circulation.

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I. Introduction to the Joys of Fluid- Flows, Fluxes, and Circulation

Laminar flows are flows in which streamlines travels in parallel layers without disruption between the layers. Each layer can be visually represented as a 2D vector field, where the field lines correspond to the streamlines of the flow. An examination of the vector field model of fluid flow reveals patterns in fluid behavior. We expect, for instance, smooth flows to be modeled by smooth streamlines and turbulent flows to appear choppy and curly-q'd. These contrasting field patterns are two extreme cases of how a field might be comprised of diverging, converging, and circulating patterns. A diverging pattern is one in which the flow appear to radiate outwards from a source or conversely, converging to a "sink". A circulation pattern is one in which the field lines tend to swirl (note: swirl doesn't necessarily imply net circulation, but we will return to this point later on). Even the most complicated of fields can be represented as some combination of these patterns. In the case of the turbulent flow, we would expect to see divergent, convergent, or circulating patterns. (Ref: Cole)

Flux and Circulation

Mathematicians have devised a clever system for evaluating the degree of a field's divergence and "swirliness." By introducing the concepts of flux and circulation of a field, we can begin to obtain a microscopic picture of what is happening within the fluid as described by the two characteristic field patterns.

What's this flux business?

Informally, flux is the amount of something crossing a surface. The flux across a hypothetical surface placed in a vector field is a reflection of the density of field lines at that location as well as the velocity distribution on the boundary of that surface. In our consideration of fluid flow, we think of flux as the amount of flow, modeled by streamlines, across a surface placed within the fluid. Positive flux corresponds to more fluid moving into the surface boundary than out, and conversely, negative flux corresponds to be more moving in than out. (*BetterExplained*)

Graphically, we can determine whether flux is positive, negative, or zero over a closed boundary by considering how many field lines enter through the boundary compared to how many exit. If the number of entering lines exceeds the number exiting, at least one field line terminates within the boundary. We take this scenario to mean that fluid enters the surface, accumulating within and resulting in positive flux. If the entering and exiting streamlines are equal, no fluid accumulates within the surface and that net flow into the surface is therefore zero. Thus we observe that net flux over a surface occurs only when streamlines terminate or originate within its boundary. This observation naturally leads the way to the conclusion that flux can be employed to "reveal" divergence or convergence of a fluid. (Cole)

What's circulation?

Just as an examination of flux is well-suited for investigating the source of the field an examination of the circulation reveals to us location of swirling currents within a fluid. Let's consider a more precise definition of circulation. Circulation is the tendency for a fluid to produce rotational motion. As mentioned previously, "swirliness" is somewhat of a cop-out for describing the tendency of current to rotate. Circulation in a fluid is the result of net angular momentum. Suppose that an object is placed in a fluid and that the circulation about that object's boundary is nonzero. This means that the object will gain angular momentum and begin to rotate. (Cole)

Intuitively, we see that angular momentum in the fluid is required for circulation. A theorem known as Kelvin's theorem states that conserved angular momentum results in constant circulation. This assertion that circulation arises and is proportional to angular momentum allows us to define an irrotational fluid as one in which there is no angular momentum, or equivalently, no circulation. This connection between circulation and angular momentum, allows us to think of circulation along a closed loop as a reflection of how "strongly" the field's vectors, if we imagine them to be force vectors, "push" along the path. (Cole)

Flux and circulation…so what?

A theorem that lays the foundation for fluid dynamics and electricity and magnetism, known as the Uniqueness Theorem, states that "a vector field F will be uniquely determined by specifying its flux and circulation." Therefore, specifications of flux and circulation may be used interchangeably with the explicit vector field representation to describe a fluid's flow. Furthermore, the fluid field may be constructed by solving the equations describing flux and circulation (Cole). In this report, we will not work out an example of back-calculating from flux and circulation to obtain the field equation, however we will work out how a field equation can be used to derive flux and circulation of the field.

II. Flux and Circulation Equations

So far we have physical and graphical descriptions of flux and circulation. Now we introduce a way to mathematically "capture" these characteristics so that they can be compared quantitatively and concretely.

The mathematical definition of flux on a simple, positively-oriented, and closed contour C is

$$\text{Net flux} = \oint_C \mathbf{F} \bullet \mathbf{N} ds$$

where **F** corresponds to the velocity vector describing the field, and **N** is the normal unit vector to the contour's surface (*BetterExplained*). Intuitively, we understand flux as dependent on the geometry, orientation, and position of the surface bounded by C and the field strength at that location. We see that mathematical definition of flux incorporates each of these factors in the calculation for flux.

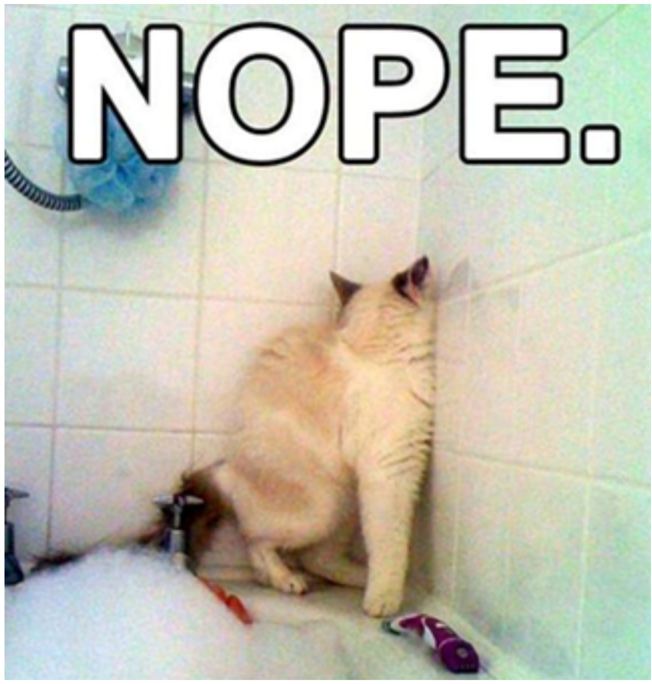
Although we will not go into depth as to why, it is important to note that only the part of the surface perpendicular experiences flux. Consequently, the dot product is use to project the flow vector onto normal vector **N** so that only the components of flow perpendicular to the surface are considered. (*BetterExplained*)

The mathematical definition of circulation along simple, positively-oriented, and closed contour C is

$$\text{Circulation} = \oint_C \mathbf{F} \bullet \mathbf{T} ds$$

where **F** corresponds to the velocity vector describing the field, and **T** is the tangent unit vector to the path C. Analogous to the use of the dot product in defining flux, circulation about a boundary depends only on the component parallel to the path because only this component "pushes" along the path, or in other words, along the tangent to the path. (*BetterExplained*)

In a perfect world, we could solve for flux and circulation over any C provided that the geometry and orientation of a boundary C and the vector field are known. It all comes down to using the formula and chugging out the answer, right?



Certain conditions that must be fulfilled in order to deploy these mathematical definitions. Firstly, in order to guarantee that you are able to take the integral of **F** over C, **F** must be continuous over an open domain D containing C. Secondly, C must be traversed in the positive direction in order for the directionality of **N** to be consistent. C must also be continuously piecewise differential in order for us to have **T**. (Tollisen)

Once these criteria are fulfilled, we can use the definition to solve for flux and circulation. However, mathematicians, like most folks, will avoid doing extra work if they can help it, and have found other strategies for calculating flux and circulation that may be more computationally straightforward to evaluate than the circulation integrals directly.

ENTER: DIV & CURL

III. Div & Curl

Mathematicians have devised a way to reflect flux and circulation as consequences of field behavior known as divergence and curl. Using an equivalent definition for flux and circulation using the new definitions of divergence and curl, we can circumvent some messiness in our calculations.

Let's chat about divergence:
The divergence through an area within a field is the rate at which flux enters and leaves a given area. We see that the mathematical definition of divergence reflects this:

$$\text{div}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$$

divergence of a field $\mathbf{F}$=P(x,y)$\mathbf{i}$+Q(x,y)$\mathbf{j}$ (*BetterExplained*)

By inspection of this definition, we see that it describes the field strength over some tiny change in position dx and dy. Thus divergence can also be thought of a reflection of how flux density varies over a microscopic area. (*BetterExplained*)

A theorem proved by Gauss known as the Divergence Theorem connects flux and divergence of a field:

$$\text{Net flux}= \oint_C \mathbf{F} \bullet \mathbf{N} ds = \iint_R div\boldsymbol{F} dxdy \qquad \text{(eq. 1)}$$

If this equation had a means to verbally express itself, it would say, "Let us imagine taking the surface and dividing it up into infinitely small pieces. If the surface is 2-dimensional, we can model each micro-surface as a tiny square with side lengths dx and dy. Then we find the microscopic divergence across each piece, which tells us about the flux density over that area. When we sum these infinitesimal pieces that comprise the entire region R bounded by C, we get the net flux over that surface. (Cole)

Curl is a useful descriptor for more than mustaches.

Curl is the circulation per unit area and reflects circulation density at a given location. Mathematically, the curl of a flow **F** in the x-y plane is defined as

$$\text{curl} = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) \qquad \text{(Cole)}$$

Analogous to Gauss's Divergence Theorem, we have Green's Theorem relating circulation and curl:

$$\text{Circulation} = \oint_C \mathbf{F} \bullet \mathbf{T} ds = \iint_R curl\boldsymbol{F} dxdy \qquad \text{(eq. 2)}$$

The expression takes the microscopic circulations along the loop C and sums them to obtain the net circulation. (Shafran)

IV. Ideal Fluid Flow

Now that we have various equations for flux and circulation of fluid flow, we begin to see how the physical features of a fluid are reflected by quantifiable characteristics. In these last few remaining pages, we will explore how the properties of ideal fluids translate to mathematical parameters.

An ideal fluid is one that is incompressible, maintaining a constant density even when moving through regions of varying pressure. Ideal fluids are also "inviscid"—which, in addition to sounding like the band name of a grindcore trio, means without viscosity. (Cole)

What are the implications of these properties?

Consider friction in an ideal fluid. A fluid with low viscosity can be alternatively described as having very low density. If molecules are more spread out, they experience and exert less drag due to jostling against one another. In an ideal fluid, we consider the effects of drag, specifically friction, negligible. (Cole)

What is the consequence of lack of friction? Were it not for friction, stirring a fluid with a spoon would be impossible, as there would be no way for streamlines to propagate the angular momentum necessary for swirling motion. We expect, if angular momentum cannot be created or conserved, that a fluid will have no vorticity. Thus we expect perfectly ideal fluids to have zero circulation! (Cole)

And what of flux? Nonzero flux is the result of mass accumulating or evacuating a region. But in an ideal fluid, density is constant throughout. No accumulation or evacuation occurs without change in density. Therefore, we expect ideal fluids to have zero flux! (Cole)

V. Ideal Fluids: zero flux & circulation

Let's explore an example of using div and curl to solve for flux and circulation.

The function f(z)=P(x,y)+iQ(x,y) is a complex representation of a velocity field modeling the laminar flow of an ideal fluid on a simply connected domain D of the complex plane. P and Q are supposed to have continuous partial derivatives throughout D. Show that given any simple closed curve C lying within D, net flux across C and the circulation around C are zero. (Adapted from Zill & Shanahan, Ch5.6, problem 31)

For any *ideal* (planar) fluid flow modeled by f(z)=P(x,y)+iQ(x,y),

$$\frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y}, \text{ and } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \qquad \text{(Zill and Shanahan)}$$

Because P(x,y) and Q(x,y) have continuous first order partial derivatives on the domain in D which loop C lies in, our system satisfies the constraints for applying eq. 1 and 2 of section 5. Thanks to the established a relationship between flux and divergence and circulation and curl, the demonstration that flux and circulation equal zero is almost trivial. But here goes…

Show that flux over C is zero:

Therefore, div= $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$

$$= \frac{\partial P}{\partial x} + -\frac{\partial P}{\partial x} \qquad \text{since } \frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y},$$

$$= 0.$$

According to Green's, net flux= $\iint_R \ div\boldsymbol{F} dxdy \ = \iint_R \ (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) dxdy = \iint_R \ (0) dxdy = 0$

So flux=0

Show that circulation along C is zero:

curl**=**

$$= \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)$$

$$= \left(-\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial x}\right) \qquad \text{since } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

=0

According to Green's, circulation=$\iint_R \ curl\boldsymbol{F}dxdy \ = \iint_R \ (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \iint_R \ (0)dxdy = 0$

So circulation=0

VI. Complex Functions and Fluid Flow

We notice that the requirement of planar ideal fluid flow, $\frac{\partial P}{\partial x}=-\frac{\partial Q}{\partial y}$ and $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, is reminiscent of the Cauchy Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. And in fact, if we consider $\overline{f(z)}$ = P(x,y) – iQ(x,y), we see that the CREs are satisfied:

$$\frac{\partial P}{\partial x}=-\frac{\partial(-Q)}{\partial y}=\frac{\partial Q}{\partial y} \quad \text{and} \quad \frac{\partial P}{\partial y}=\frac{\partial(-Q)}{\partial x}=-\frac{\partial Q}{\partial x}$$

A complex function describing ideal fluids has an analytic complex conjugate. (Olver)
Interesting… So if we take a circulation integral of analytic $\overline{f(z)}$ with continuous first-order partial derivatives over a C lying in an open, simply connected domain on which $\overline{f(z)}$ is analytic, we expect by the Cauchy-Goursat Theorem, the value of $\oint_C \bar{\mathbf{F}}dz$ to be *zero*!

According to Zill and Shanahan (excerpted directly from Ch5.6)

$\oint_C \bar{\mathbf{F}}dz = \oint_C (\mathrm{P} - \mathrm{i\,Q})(\mathrm{dx} + \mathrm{i\,dy})$

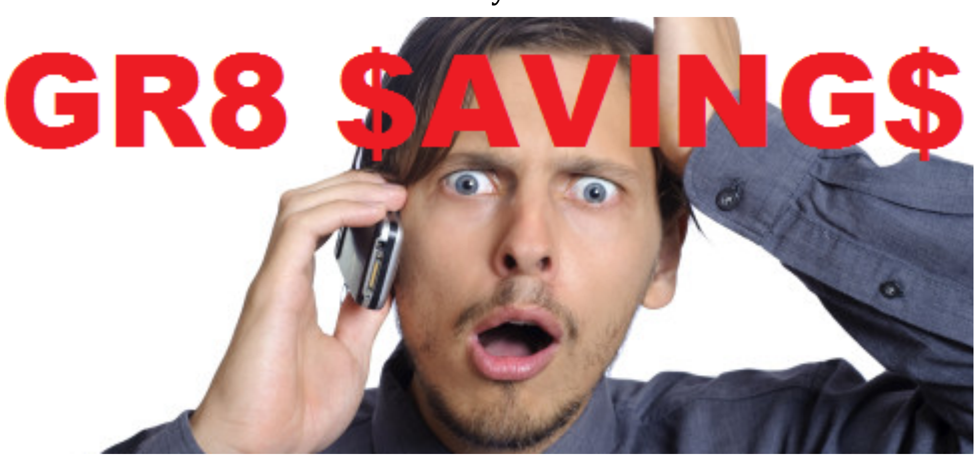
$= (\oint_C \mathrm{P\,dx} - \mathrm{i\,Q\,dy}) + \mathrm{i}\,(\oint_C \mathrm{P\,dy} - \mathrm{i\,Q\,dx})$

$= (\oint_C \mathrm{P\,dx} - \mathrm{i\,Q\,dy}) + \mathrm{i}\,(\oint_C \mathrm{P\,dy} - \mathrm{i\,Q\,dx})$

$=(\oint_C \mathbf{F} \bullet \mathbf{N}ds) + \mathrm{i}\,(\oint_C \mathbf{F} \bullet \mathbf{T}ds)$

=(flux) + i (circulation)

Thus our result that $\oint_C \bar{\mathbf{F}}dz = 0$ implies that both flux and circulation are zero, as we expect of an ideal fluid. We also notice that by simply evaluating $\oint_C \bar{\mathbf{F}}dz$, we get a two for one deal, solving for flux and circulation simultaneously!



VII. Conclusion

We have now demonstrated a few methods for evaluating flux and circulation:

1) Using Net flux= $\oint_C \mathbf{F} \bullet \mathbf{N} ds$ and Circulation= $\oint_C \mathbf{F} \bullet \mathbf{T} ds$
2) Using div and curl definitions for
3) Modeling the vector field as a complex function f(z) then evaluating $\oint_C \bar{\mathbf{F}} dz$

In our investigation of these methods, we saw how physical phenomena inspire mathematical models, and how those representations reflect intuitive understandings of what transpires in the tangible universe. Most often, a variety of approaches, qualitative and quantitative, must be employed to solve a problem. Clear to any student of mathematics, is the utility of equivalence in solving problems, that is, in being able to express one concept in multiple forms, giving way to perhaps simpler strategies for obtaining a solution. Finally, our discussion of complex functions and ideal fluid flow illustrates how shuttling between alternate forms for the “same thing” enables, for instance, computation performed in the world of the imaginary to afford solutions even in the realm of the real.

References

Cole. *Hydrodynamics Irrotational Flows and Flux*. N.p.: UC Davis, n.d. PDF.

Olver, Peter J. "Complex Analysis." N.p.: U of Minnesota, 2012. N. pag. Web.

Shafran. "Green's Theorem." *Green's Theorem*. N.p., n.d. Web. 11 Apr. 2014.

Tollisen, Gregory. CAE Math Specialist. Consulted on 4/14/2014 and 4/29/2014.

"Vector Calculus: Understanding Circulation and Curl." *BetterExplained*. N.p., n.d. Web. 11 Apr. 2014.

"Vector Calculus: Understanding Divergence." *BetterExplained*. N.p., n.d. Web. 11 Apr. 2014.

"Vector Calculus: Understanding Flux." *BetterExplained*. N.p., n.d. Web. 11 Apr. 2014.

Zill, Dennis G., and Patrick D. Shanahan. "Chapter 5.6 Applications." *A First Course in Complex Analysis with Applications*. Sudbury, MA: Jones and Bartlett, 2009. N. pag. Print.