An Application of the Inverse Laplace Transform

Madyson Cassidy

April 30, 2014

Abstract

Laplace transforms are often applied to determine how to solve problems involving partial differential equations. I investigate how to use several theorems from Zill & Shanahan to evaluate the expression

$$f(x,t) = \mathscr{L}^{-1}\left\{\frac{\sinh xs}{(s^2+1)\sinh s}\right\}.$$

We find that $f(x,t) = \dfrac{\sin x \sin t}{\sin 1}$ as long as the aforementioned theorems and hypotheses of these theorems are assumed to be true. This result is not surprising given that $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$, but it does not necessarily follow that $\mathscr{L}^{-1}\left\{\dfrac{\sinh xs}{\sinh s}\right\} = \dfrac{\sin x}{\sin 1}$. This paper is an examination of the solution, $f(x,t)$, to the Laplace inverse transform of $F(x,s)$, namely $\dfrac{\sinh xs}{(s^2+1)\sinh s}$.

1 Theorems

Theorem 6.7.3 [1] shows the principal value of an integral written as

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma+i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds,$$

where $F(s)$ is come function of a complex number s and the limit of integration is on the vertical line $Re(s) = \gamma$ with all of the real parts on the left half plane. This integral is known as the **Bromwich contour integral** (see Figure 1 below).

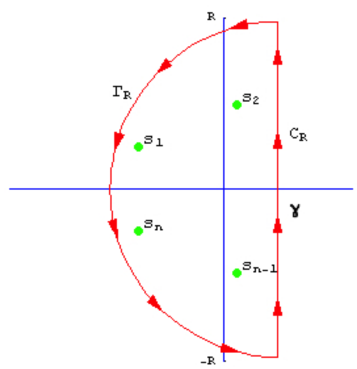

Figure 1: Bromwich Contour

The contour from the previous theorem is used in the proof of Theorem 6.7.4. This is considered to be a Laplace transform with a finite number of poles to the left of the vertical line presented previously, or $Re(s) = \gamma$, and in the contour of an arc from $\gamma + iR$ to $\gamma - iR$. By assuming the hypothesis of Theorem 6.7.4 is true, we know $sF(s)$ is bounded on the arc as $R \to \infty$ so that we can apply Theorem 6.7.4, written as

$$\mathscr{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} Res\Big(e^{st}F(s), s_k\Big).$$

The function $f(x,t)$ results from the Laplace transform of $F(x,s)$ as long as $t > 0$. For $t < 0$, the Bromwich contour integral is equal to zero because the function is analytic everywhere within and on that contour. In other words, a contour that is the right half of a circle including the line $Re(s) = \gamma$ from $\gamma + iR$ to $\gamma - iR$ is the contour of $t < 0$ where no poles exist in or on the contour. On the other hand, when $t > 0$, the contour is the left half of a circle from $\gamma + iR$ to $\gamma - iR$ including the line $Re(s) = \gamma$ that contains the poles of $F(x,s)$.

2 An Example

For example, consider $F(s) = \dfrac{1}{s}$. To find the inverse Laplace transform of $F(s)$, we must determine the poles of $F(s)$. Clearly a pole appears at $s = 0$. Thus, we must find $Res\left[\dfrac{e^{st}}{s}, 0\right]$. We know that when $F(s) = \dfrac{g(s)}{h(s)}$ and $F(s)$ has a pole of order one, then $Res[e^{st}F(s), s_0] = \dfrac{e^{s_0t}g(s_0)}{h'(s_0)}$. It follows that if $e^{st}g(s) = e^{st}$ and $h(s) = s$, and $s_0 = 0$,

$$Res\left[\frac{e^{st}}{s}, 0\right] = \frac{e^{0t}}{1} = 1.$$

Hence, we see that $\mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$.

3 Determining the Inverse Laplace Transformation

In order to use Theorem 6.7.4, we must determine the residues of the poles described by $e^{st}F(x,s)$ for our $F(x,s) = \dfrac{\sinh xs}{(s^2+1)\sinh s}$. Since poles occur at $s = n\pi i, i, -i$ $(n \in \mathbb{Z})$ we must find $Res[e^{st}F(x,s); n\pi i], Res[e^{st}F(x,s); i]$, and $Res[e^{st}F(x,s); -i]$.

For $Res[e^{st}F(x,s);n\pi i]$, let $g(x,s)=\dfrac{e^{st}\sinh xs}{s^2+1}$ and $h(x,s)=\sinh s$. Then,

$$Res(e^{st}F(x,s);n\pi i)=\frac{g(x,n\pi i)}{h_s(x,n\pi i)}=\frac{\frac{e^{n\pi it}\sinh n\pi ix}{(n\pi i)^2+1}}{\cosh n\pi i}=0 \text{ for } n\in\mathbb{Z}.$$

Unfortunately this is not the case for all values of x. In our case, x is a fixed parameter that could potentially be any (real or imaginary) number. Thus, to get zero as our solution to the above residue, x must be an element of the integers so that $\sinh n\pi ix=0$. In this case, $\sinh n\pi ix=0\Leftrightarrow\dfrac{e^{n\pi ix}-e^{-n\pi ix}}{2}=0\Leftrightarrow e^{2n\pi ix}-1=0\Leftrightarrow e^{2n\pi ix}=1$ which occurs when $x\in\mathbb{Z}$. Answers for $x\notin\mathbb{Z}$ vary greatly.

For $Res(e^{st}F(x,s);i)$, let $g(x,s)=\dfrac{e^{st}\sinh xs}{(s+i)\sinh s}$ and $h(x,s)=s-i$. Then,

$$Res(e^{st}F(x,s);i)=\frac{g(x,i)}{h_s(x,i)}=\frac{\frac{e^{it}\sinh ix}{(i+i)\sinh i}}{1}=\frac{ie^{it}\sin x}{2i^2\sin 1}=\frac{-ie^{it}\sin x}{2\sin 1}$$

since $\sinh ix=i\sin x$ and $\sinh i=i\sin 1$.

For $Res(e^{st}F(x,s);-i)$, let $g(x,s)=\dfrac{e^{st}\sinh xs}{(s-i)\sinh s}$ and $h(x,s)=s+i$. Then,

$$Res(e^{st}F(x,s);i)=\frac{g(x,-i)}{h_s(x,-i)}=\frac{\frac{e^{-it}\sinh -ix}{(-i-i)\sinh -i}}{1}=\frac{-ie^{-it}\sin x}{-2i^2-\sin 1}=\frac{ie^{-it}\sin x}{2\sin 1}$$

since $\sinh -ix=-i\sin x$ and $\sinh -i=-i\sin 1$.This result is similar to the previous residue solution because we found the conjugate of the previous residue which should simply be the conjugate of the solution, which it is.

Therefore we find that

$$\begin{aligned}f(x,t)=\mathscr{L}^{-1}\{F(x,s)\}&=\sum_{k=1}^{3}Res(e^{st}F(x,s),s_k)=0+\frac{-ie^{it}\sin x}{2\sin 1}+\frac{ie^{-it}\sin x}{2\sin 1}\\&=\frac{i\sin x}{\sin 1}\left(\frac{-(e^{it}-e^{-it})}{2}\right)=\frac{-i\sin xi\sin t}{\sin 1}=\frac{\sin x\sin t}{\sin 1}\end{aligned}$$

References

[1] Shanahan, P. D. & Zill, D. G. *Complex Analysis: A First Course with Applications* 2013: Jones & Bartlett Learning.