Complex Analysis

Class 23: **Wednesday March 24**

SUMMARY Applications of Cauchy-Goursat theorem
CURRENT READING Saff & Snider, §4.3

Previously we learned that the **Cauchy-Goursat Theorem** implies that,
given f is continuous in a domain D if any one of the following is true, then so are the others:

(a) f has an antiderivative in D, called $F(z)$, such that $F'(z) = f(z)$

(b) the integrals of $f(z)$ along contours lying entirely in D extending from z_1 to z_2 all have the same value, namely $F(z_2) - F(z_1)$

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero

$\boxed{\text{EXAMPLES}}$

Evaluate the following integrals. In each case sketch the contour you used and explain how you evaluated the integral, i.e. what idea are you using. Also answer the questions.

1. $\displaystyle\int_{-i}^{i} \frac{dz}{z}$

Does the integral value change depending on the path from $-i$ to i?
Does $1/z$ have an antiderivative? In what domain?

2. $\displaystyle\oint_{|z|=1} \frac{dz}{z}$

Does the integral value change depending on what direction one takes along $|z| = 1$?
What would the value of the integral be if the contour were traversed twice in the clockwise direction?

Loop
A **loop** or simple closed contour is said to be traversed in the **positive sense** if an observer traversing the contour in that direction would always see the *interior* of the contour on their left.

Simply-Connected and Multiply-Connected Domains

A **simply-connected** domain D is one such that *every* simple closed contour (i.e. loop) lying in D encloses only points of D

A domain which is *not* simply-connected, is called **multiply-connected**.

$\boxed{\text{GROUPWORK}}$

Sketch and classify the following domains as **simply-connected** or **multiply-connected**

(a) $|\text{Im } z| < 1$

(b) $1 < |z| < 2$

(c) $\mathbf{C} \setminus \{ \text{Re}(z) > 0 \cap \text{Im } (z) = 0\}$

(d) $|z| < 4 \setminus \{|z - i| < .5 \cup |z + i| < .5\}$

FACT: Simply-connected domains have the property that every loop in D can be continuously deformed in D to a single point.

FACT: $\int_C f(z) dz \equiv 0$ if C is a point.

Deformation Invariance Theorem

Let f be an analytic function in a domain D containing loops Γ_0 and Γ_1. If these loops can be continuously deformed into each other by passing through points only in D, then

$$\oint_{\Gamma_0} f(z)\, dz = \oint_{\Gamma_1} f(z)\, dz$$

$\boxed{\textbf{Exercise}}$

Evaluate the integral below, where Γ is SOME RANDOM **closed** contour (i.e. a loop).

$$\oint_{\Gamma} \frac{dz}{z - z_0}$$

Does it matter where $z_0 \in \mathbf{C}$ is?