Complex Analysis

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Class 17: **Monday March 1**

SUMMARY Complex Exponents z^c and c^z
CURRENT READING Saff & Snider, §3.5
HOMEWORK Saff & Snider, Section 3.5 # 1, 2, 3, 4, 7

Roots of Complex Numbers (Reprise)
From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in Z$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right)$$

But

$$\begin{aligned} \exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln|z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in Z) \\ &= |z|^{1/n}\exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right) \\ &= |z|^{1/n}\exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z \end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where k is restricted to $0, 1, 2, \ldots, n-1$. Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when k can be any integer and n is fixed?] What do you think happens i f we try and raise a complex number to something besides an integer or rational number? How will we deal with **complex exponents**?

Complex Exponents
If $z \neq 0$ and $c \in \mathcal{C}$, the function z^c is defined as

$$z^c = \exp(\log z^c) = \exp(c \log z)$$

Since $\log z$ is a multi-valued function, z^c will have multiple values. How many values depends on the nature of c.

$$z^c = \begin{cases} z^{n/m} & \text{if } c \text{ is rational, i.e. } n/m & \text{finite number of values (m)} \\ z^n & \text{if } c = n, \text{ where } n \text{ is an integer} & \text{single value} \\ z^c & \text{all other complex numbers} & \text{infinite number of values} \end{cases}$$

EXAMPLE

Show that i^i is purely real.

GroupWork

Compute the following:

(a) $(0.5 - \frac{\sqrt{3}}{2}i)^3 =$

(b) $(-1)^{2/3} =$

(c) $(1+i)^{1-i} =$

Derivatives of z^c and c^z

If you choose a branch of z^c which is analytic on an open set, then

$$\frac{d}{dz}(z^c) = cz^{c-1}$$

where the branch of the log used in evaluating z^c is the same branch used in evaluating z^{c-1}
Similarly, we can define the *complex exponential function with base c*

$$c^z = \exp(z \log c)$$

If a single value of c is chosen, then c^z is an entire function such that

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\exp(z \log c) = c^z \log c$$

EXAMPLE

If $f(z) = (1+i)^z$, Find $f'(1-i)$ (Use $\mathcal{L}_0$)

If $g(z) = z^{(1-i)}$, Find $f'(1+i)$ (Use $\mathcal{L}_0$)