Complex Analysis

Math 312 Spring 2004 — MWF 3:30-4:25

 — Fowler 316

Name: ______________________________

TEST 1: Friday, March 5, 2004

Directions: Read *all* 3 problems first before answering any. Notice the HINTS on each problem. You may have access to any notes or the textbook. This is a one hour test. You have 90 minutes to complete it.

No.	Score	Maximum
1		40
2		30
3		30
Total		100

1. *[40 pts. total]* **Mapping.** We want to understand the implications on the location and orientation of the branch cut of the Principal Branch of the Complex Logarithm function when the argument changes from Log (z) to Log $(Az + B)$ where A and B are complex numbers.
HINT: The principal branch cut is at $D = \{z : \text{Re}(z) \leq 0 \cap \text{Im}(z) = 0\}$.

(a) *[5 pts]* Sketch the location and orientation of the branch cut of Log (z) under the mapping $w = f_1(z) = z + z_0$ where $z_0 \in \mathcal{C}$.

(b) *[5 pts]* Sketch the location and orientation of the branch cut of Log (z) under the mapping $w = f_2(z) = e^{i\theta} z$ where $\theta \in \mathcal{R}$.

(c) *[5 pts]* Sketch the location and orientation of the branch cut of Log (z) under the mapping $w = iz + 1$ where $\theta \in \mathcal{R}$.

(d) *[5 pts]* Sketch the location and orientation of the branch cut of Log $(iz + 1)$. [HINT: think about how part (c) and (d) are related questions, but different!]

(e) *[10 pts]* Find a function Log $(Az + B)$ which has its branch cut located at $D = \{z = x + iy : y = x \cap x \geq 1\}$

(f) *[10 pts]* Is it possible to find a single-valued branch of $\log(z)$, i.e. $\mathcal{L}_\alpha(z)$ which has its branch cut at the same exact location and orientation as D from part (e)? **Why, or why not?** If possible, write down a function involving $\mathcal{L}_\alpha$ which has the same branch cut D.

2. *[30 pts.]* **Algebra of Complex Numbers.**

Consider $\cos(z) = b$ where $b \in \mathcal{C}$. Therefore, $z = \cos^{-1}(b) = \arccos(b)$

HINT: $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

(a) *[10 pts]* Show that $z = \arccos(b) = -i\log(b \pm \sqrt{b^2 - 1})$

(b) *[10 pts]* When $b = 0$, use information from part **(a)** to help you evaluate $z = \arccos(0)$. **Indicate the location of your solutions in the Complex Plane.**

(c) *[10 pts]* When $b = 2$, use information from part **(a)** to help you evaluate $z = \arccos(2)$ **Indicate the location of your solutions in the Complex Plane.**

3. *[30 pts. total]* **Cauchy-Riemann Equations, Harmonic Conjugates**. Consider an analytic function $f(z) = u(x,y) + iv(x,y)$. We want to show that the set of implicitly-defined curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal to each other at their point of intersection. Note that c_1 and c_2 are real constants. **HINT:** two curves are perpendicular whenever the product of their slopes equals -1.

(a) *[10 pts]* For $f(z) = Az + B$ where A and B are complex constants, find $u(x,y)$ and $v(x,y)$. Show that the slopes of the implicit curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are perpendicular to each other whenever they intersect.

(b) *[10 pts]* For $f(z) = z^2$ find $u(x,y)$ and $v(x,y)$ and show that the slopes of the implicit curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are perpendicular whenever they intersect.

(c) *[10 pts]* Building on your answers in (a) and (b), use implicit differentiation and the Cauchy Riemann Equations to prove the general principle that for an analytic function $f(z) = u(x,y) + iv(x,y)$ the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ are orthogonal (i.e. perpendicular at a general point in the xy-plane).